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High-Order Fully Discrete Energy Diminishing Evolving Surface Finite Element Methods for a Class of Geometric Curvature Flows

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Abstract. This article concerns the construction of high-order energy-decaying numerical methods for gradient flows of evolving surfaces with curvature-dependent energy functionals. The semidiscrete evolving surface finite element method is derived based on the calculus of variation of the semidiscrete surface energy functional. This makes the semidiscrete problem naturally inherit the energy decay structure. With this property, the semidiscrete problem is furthermore formulated as a gradient flow system of ODEs. The averaged vector-field collocation method is used for time discretization of the ODEs to preserve energy decay at the fully discrete level while achieving high-order accuracy in time. Extensive numerical examples are provided to illustrate the accuracy and energy diminishing property of the proposed method, as well as the effectiveness of the method in capturing singularities in the evolution of closed surfaces.

AMS subject classifications: 65M60, 65M70, 35R01

Key words: Gradient flow, evolving surface, curvature, energy decay, evolving surface, finite element method, averaged vector-field collocation.

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1 Introduction

Let Γ^0 be a $(d-1)$-dimensional hypersurface in $\mathbb{R}^d$, with $d\geq 2$. For any smooth embedding $\phi:\Gamma^0\to\mathbb{R}^d$, we denote by $\Gamma[\phi]$ the hypersurface as the image of ϕ, i.e.,

$$\Gamma[\phi]=\{\phi(p):p\in\Gamma^0\}.$$

We consider the L^2 gradient flow of hypersurfaces $\Gamma[u(\cdot,t)]$ determined by a family of embeddings $u(\cdot,t):\Gamma^0\to\mathbb{R}^d$, with $t>0$, associated to an energy functional

$$E[u]=\int_{\Gamma[u]} f(\kappa)\mathrm{d}\Gamma, \tag{1.1}$$

where $\kappa=H\nu$ is the vector-valued curvature, with H and ν denoting mean curvature and unit normal vector on the surface. The initial surface is simply Γ^0, determined by the initial condition $u(p,0)=p$ for $p\in\Gamma^0$. The evolution equation of this problem can be written as

$$\begin{cases} \partial_t u=-E'[u]\circ u, \\ u|_{t=0}=\mathrm{Id}, \end{cases} \tag{1.2}$$

where $\mathrm{Id}:\mathbb{R}^d\to\mathbb{R}^d$ is the identity map, and $E'[u]:\mathbb{R}^d\to\mathbb{R}^d$ is the function under the Gateaux derivative of the energy functional $E[u]$, defined on the surface $\Gamma[u]$ satisfying the following relation:

$$\int_{\Gamma[u]} E'[u]v\mathrm{d}\Gamma=\frac{\mathrm{d}}{\mathrm{d}\epsilon}E[u+\epsilon v\circ u]\bigg|_{\epsilon=0}, \quad \forall v\in C^\infty(\Gamma[u]). \tag{1.3}$$

Equations derived from (1.2), with curvature-dependent energy functionals in the form of (1.1), are referred to as curvature flows, which are often used to model morphology of microstructure in material sciences [12], biomembranes [6], cell movement and chemotaxis [26,34], etc. Different energy functional $E[u]$ leads to different evolution equations. In particular, the following curvature flows are widely considered in the literature (for example, see [12,15,22]):

- Curve shortening flow: $d=2$ and $f(\kappa)=1$. In this case, the evolution equation of u can be written as
$$\partial_t u=(\partial_\Gamma^2\mathrm{Id})\circ u,$$
where ∂_Γ denotes the tangential derivative along the curve $\Gamma[u]$.