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Optimization with Least Constraint Violation

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Abstract. Study about theory and algorithms for nonlinear programming usually assumes that the feasible region of the problem is nonempty. However, there are many important practical nonlinear programming problems whose feasible regions are not known to be nonempty or not, and optimizers of the objective function with the least constraint violation prefer to be found. A natural way for dealing with these problems is to extend the nonlinear programming problem as the one optimizing the objective function over the set of points with the least constraint violation. Firstly, the minimization problem with least constraint violation is proved to be an Lipschitz equality constrained optimization problem when the original problem is a convex nonlinear programming problem with possible inconsistent constraints, and it can be reformulated as an MPCC problem; And the minimization problem with least constraint violation is relaxed to an MPCC problem when the original problem is an nonlinear programming problem with possible inconsistent non-convex constraints. Secondly, for nonlinear programming problems with possible inconsistent constraints, it is proved that a local minimizer of the MPCC problem is an M-stationary point and an elegant necessary optimality condition, named as L-stationary condition, is established from the classical optimality theory of Lipschitz continuous optimization. Thirdly, properties of the penalty method for the minimization problem with the least constraint violation are developed and the proximal gradient method for the penalized problem is studied. Finally, the smoothing Fischer-Burmeister function method is constructed for solving the MPCC problem related to minimizing the objective function with the least constraint violation. It is demonstrated that, when the positive smoothing parameter approaches to zero, any point in the outer limit of the KKT-point mapping is an L-stationary point of the equivalent MPCC problem.

AMS subject classifications: 90C30

Key words: Nonlinear programming, least constraint violation, L-stationary point, MPCC, smoothing function.

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1 Introduction

For studying an ordinary nonlinear optimization problem, a basic assumption is that feasible region of the optimization problem is nonempty. Many important theoretical issues are well studied for an optimization problem under this assumption. For example, optimality theory and sensitivity analysis are two main theoretical topics. Optimality theory consists of necessary optimality conditions and sufficient optimality conditions. Sensitivity analysis studies continuity properties of the optimal value and the solution mapping when the optimization is perturbed. For nonlinear programming, for a local minimizer, the first-order necessary optimality conditions and the second-order optimality conditions can be developed under certain constraint qualifications, and the second-order sufficient optimality conditions imply the second-order growth condition, see for instance the famous textbook [13]. For nonlinear programming, a series of stability results were obtained by Robinson, see [14–16]. Bonnans and Shapiro [3] established the optimality theory and the stability theory for general optimization problems, including problems whose decision variables are infinite dimensional, nonlinear semidefinite programming problems and other conic optimization problems.

However, when the feasible set is empty or the constraints are inconsistent, infeasibility detection is an important issue for algorithmic design. Many numerical algorithms have been proposed to find infeasible stationary points; namely, stationary points for minimizing certain infeasibility measure. Byrd, Curtis and Nocedal [4] presented a set of conditions to guarantee the superlinear convergence of their SQP algorithm to an infeasible stationary point. Burke, Curtis and Wang [5] considered the general program with equality and inequality constraints, and proved that their SQP method has strong global convergence and rapid convergence to the KKT point, and has superlinear/quadratic convergence to an infeasible stationary point. Recently, Dai, Liu and Sun [7] proposed a primal-dual interior-point method, which can be superlinearly or quadratically convergent to the Karush-Kuhn-Tucker point if the original problem is feasible, and can be superlinearly or quadratically convergent to the infeasible stationary point when the problem is infeasible.

These algorithms can find a stationary point of the infeasibility measure, which have nothing to do with the objective function of the problem. In practice, there are many important problems that we need to find minimizers of the objective function over the points with the least constraint violation. A natural way to deal with such problems is to extend the constrained optimization problem as the one that optimizes the objective function over the set of points with least constraint violation. When the feasible region is nonempty, the set of points with least constraint violation coincides with the feasible region of the constrained optimization problem and hence the extended constrained optimization problem coincides with the original problem.

Now we give a formal definition of infeasibility measure of an nonlinear programming problem. Suppose that the nonlinear programming problem is of the following