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Energy Stable Arbitrary Order ETD-MS Method for Gradient Flows with Lipschitz Nonlinearity

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Abstract. We present a methodology to construct efficient high-order in time accurate numerical schemes for a class of gradient flows with appropriate Lipschitz continuous nonlinearity. There are several ingredients to the strategy: the exponential time differencing (ETD), the multi-step (MS) methods, the idea of stabilization, and the technique of interpolation. They are synthesized to develop a generic k^{th} order in time efficient linear numerical scheme with the help of an artificial regularization term of the form $A\tau^k \frac{\partial}{\partial t}\mathcal{L}^{p(k)}u$ where $\mathcal{L}$ is the positive definite linear part of the flow, τ is the uniform time step-size. The exponent $p(k)$ is determined explicitly by the strength of the Lipschitz nonlinear term in relation to $\mathcal{L}$ together with the desired temporal order of accuracy k. To validate our theoretical analysis, the thin film epitaxial growth without slope selection model is examined with a fourth-order ETD-MS discretization in time and Fourier pseudo-spectral in space discretization. Our numerical results on convergence and energy stability are in accordance with our theoretical results.

AMS subject classifications: 65M12, 65M70, 65Z05

Key words: Gradient flow, epitaxial thin film growth, exponential time differencing, long time energy stability, arbitrary order scheme, multi-step method.

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1 Introduction

Many natural and engineering processes are gradient flows in the sense that the time evolution of the system is in the direction of decreasing certain energy functional associated with the state of the system. They have a wide range of applications in materials

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science, fluid dynamics [2,4,7,13,17,23,35,53] as well as in geometry (geometric flows) and PDEs (optimal transportation) [3] among many others. The evolution of these gradient flows could be complicated. Efficient and accurate numerical methods are highly desirable, especially in the generic case of the absence of solution formula. In addition, the evolution process could be long before it settles to certain equilibrium state(s), see for instance [34]. Hence, it is of great importance to design numerical methods that inherits the energy law of the gradient flow, even if in a slightly modified form, if one is interested in the long evolution process such as the coarsening process associated with many phase field models at the large system size regime.

An abundant work exists in the development of (energy) long-time stable schemes and numerical simulation of various gradient flows arising in material science and fluid dynamics using convex splitting, truncation, SAV, and IEQ method among others, see for instance [18, 19, 22, 43–47, 50], and the references therein. Exponential time differencing (ETD) is a very appealing time discretization method which achieves its high-order accuracy in time with exact treatment of the linear part [5, 15, 16] together with Duhammel's principle applied to the nonlinear term. The introduction of integrating factor gives rise to a nonlinear integral term. There are two popular approaches in approximating the nonlinear part: Runge-Kutta (RK) method [12, 25, 26] and multi-step (MS) method [24, 26, 27]. Abundant applications of these two methods to various gradient models can be found in [1, 8, 9, 11, 14, 28, 30–32, 52] among others. The approximation of the nonlinear term is usually explicit in order to preserve the efficiency of the ETD method. While it is relatively straightforward to construct numerical schemes of arbitrary order formally via RK or MS method [26], the energy stability of the algorithms are nontrivial since the explicit treatment of the nonlinear term induces instability, and the existing works involving rigorous energy stability analysis for high accuracy scheme are limited. The authors in [29] proved the energy stability for the first order ETD scheme for the thin film epitaxial growth model without slope selection (NSS). In a recent work, the idea of stabilization was utilized to develop a second order in time ETD-MS method that is energy stable with the aid of a judiciously chosen stabilizing term of matching order for the NSS equation [8]. This is the first second-order in time work to provide both energy stability and convergence results for the NSS model theoretically. This stabilizing idea has been developed further in [9] for a third order in time ETD-MS energy stable scheme for the NSS model, and to even higher orders without detailed proof. Another third-order stabilized energy stable ETD-MS scheme proposed in [11] to approximate the NSS equation also gave a rigorous energy stability analysis, in which a different form of stabilized term $A\tau^2\Delta_N^2(u^{n+1}-u^n)$ was added with a ϵ dependent stabilizing coefficient $A=\mathcal{O}(\epsilon^{-2})$ to guarantee energy stability. The purpose of this manuscript is to present a systematic approach to construct ETD-MS based energy stable schemes of arbitrary order in time with the help of an appropriate stabilizing term for a class of gradient flows on a Hilbert space with a positive linear part and a mild nonlinear part satisfying Lipschitz condition in some suitable sense. In particular, our result cover the NSS model as a special case. More importantly, we can make the stabilizing coefficient A independent of