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A Class of Nonlinear Degenerate Parabolic Equations Not in Divergence Form*

Cao Chun-ling, Cheng Xiao-lin and Gao Wen-jie
(*Institute of Mathematics, Jilin University, Changchun, 130012, China*)

Abstract: This paper is devoted to the study of a class of singular nonlinear diffusion problem. The existence and uniqueness of solutions are obtained. Moreover, some properties of solutions such as blow-up property etc. are also discussed.

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1 Introduction

In this paper, the following initial and boundary value problem is considered:

$$u_t = u^\sigma(\Delta_p u + f(x, t, u)), \qquad (x, t) \in \Omega_T, \tag{1.1}$$

$$u(x, t) = 0, \qquad (x, t) \in \partial\Omega \times (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $\Omega_T = \Omega \times (0, T)$, $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian, $\sigma \geq 1$, $f : \Omega \times (0, T) \times R \to R$ is a nonnegative differentiable function such that $|f(x, t, u)| \leq g(u)$, where $g \in C^1(R)$ (see [1]), and u_0 satisfies the following condition:

(H) $0 \leq u_0 \in C(\bar{\Omega}) \cap W^{1,p}(\Omega)$, $u_0 = 0$ on $\partial\Omega$.

Since (1.1) degenerates whenever $u = 0$ or $\nabla u = 0$, the problem does not admit classical solutions in general. Thereofore we need to consider weak solutions. Moreover, only nonnegetive solutions are considered.

The problem we are studying arises in biological (see [2]) and astrophysical (see [3], [4]) context. Similiar problems can be found in [5]–[7]. (1.1) also arises in some models describing physical phenomena. For example, when $\sigma = 0$, (1.1) is the general evolutional p-Laplacian. When $\sigma \in (0, 1)$, (1.1) can be written in divergence form after changing variables (provided $m(p-1) - 1 \neq 0$) by setting

$$u = m^{m/[m(p-1)-1]} v^m, \qquad m = 1/(1-\sigma),$$

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which gives the filtration equation for v (see [8]–[10]):

$$v_t = \mathrm{div}(|\nabla v^m|^{p-2}\nabla v^m) + f(v).$$

Since (1.1) may degenerate at points where $u = 0$ or $\nabla u = 0$, we consider weak solutions in this paper. Many authors have studied (1.1) with $p = 2$ (see [5]–[7], [11]–[15] and the references therein), for instance, nonuniqueness of solutions for the case $p = 2$ and $\sigma = 1$ were studied by Dal Passo and Luckhaus[5] and independently by Ughi[7]. Recently, the authors of [16] dicussed the case $p > 2$ and constructed solutions in the sense of distribution. The author of [17] discussed the global existence of solutions. In this paper, we consider the problem (1.1)–(1.3) for general case $p > 1$. As shown in [5], for the case $p = 2$ the discussion of uniqueness of solutions can be easily carried out, but for the p-Laplacian it is difficult to get its uniqueness of solutions. To overcome that difficulty, we have to redefine solutions. As seen below, by combining the special construction of the equation (1.1) we present a different definition of solutions from [16]. Although, it seems to be not natural and a little complex, the uniqueness of solutions are guaranteed. Fortunately, we can also prove the existence of solutions. Moreover, some properties of solutions, such as blow-up property etc. are also discussed in this paper.

Before giving the concept of solutions, let us first define the support of a nonnegative function $\omega : \Omega \to [0, \infty)$:

$$\mathrm{supp}\omega = \overline{\Big\{x \in G;\ \lim_{\rho\to 0^+} \frac{\mu(G \cap B_\rho(x))}{\mu(B_\rho(x))} > 0\Big\}},$$

where

$$G = \{x \in \Omega;\ \omega(x) > 0\}, \qquad B_\rho(x) = \{y \in \Omega;\ |x - y| < \rho\},$$

and $\mu(E)$ denotes the Lebesgue measure of a set E in $\mathbf{R}^N$. It is easy to see that if $\omega \in C(\Omega)$, then $\mathrm{supp}\omega = \bar{G}$. Denote $\tilde{\Omega}$ and Ω^ρ for $\rho > 0$ by

$$\tilde{\Omega} = \{x \in \Omega;\ u_0(x) > 0\}, \qquad \Omega^\rho(u_0) = \{x \in \tilde{\Omega};\ \mathrm{dist}(x, \partial\Omega) > \rho\}.$$

Definition 1.1 *A nonnegative function u is called a weak solution of* (1.1)–(1.3), *if*

(1) $u \in L^\infty(\Omega_T) \cap L^p(0, T;\ W_0^{1,p}(\Omega))$ *with* $u_t \in L^2(\Omega_T)$;

(2) *for any* $\varphi \in C_0^\infty(\Omega_T)$, *there holds*

$$\iint_{\Omega_T} [-u\varphi_t + u^\sigma|\nabla u|^{p-2}\nabla u\nabla\varphi + \sigma u^{\sigma-1}|\nabla u|^p\varphi - f(x, t, u)\varphi]\mathrm{d}x\mathrm{d}t = 0;$$

(3) $\mathrm{supp}(\,\cdot\,, t) = \mathrm{supp}u_0(\,\cdot\,)$ *a.e. in* $(0, T)$, *and for any* $\rho > 0$ *sufficiently small, there exists a positive constant* $c = c(\rho)$ *depending on* ρ *such that* $u \geq c$ *a.e. in* $\tilde{\Omega}_\rho \times (0, T)$;

(4) $\displaystyle\lim_{t\to 0^+} \int_\Omega |u(x, t) - u_0(x)|\mathrm{d}x = 0.$

If $T = \infty$, u is called a global solution. We are to prove the local existence of solutions. We always denote by $u(x, t; u_0)$ or $u(t)$ the solution with initial value u_0.

The main results of this paper are the following theorems.

Theorem 1.1 (Local existence and uniqueness) *Let $p > 1$, $\sigma \geq 1$, and assume that u_0 satisfies* (H). *Then there exists a positive constant $T^* = T^*(u_0)$ such that the problem* (1.1)–(1.3) *in Ω_{T^*} admits a unique local solution.*