COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
25(4)(2009), 349–360

Noncommutative Multi-Solitons in a Modified Chiral Model in 2+1 Dimensions*

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Communicated by Yin Jing-xue

Abstract: Multi-soliton configurations of a Moyal-type noncommutative deformed modified 2+1 chiral model have been constructed by dressing method several years ago. These configurations have no-scattering property. By making a second-order pole in the dressing ansatz, two-soliton configurations with genuine soliton-soliton interaction were constructed after that. We go on in this paper to construct a large family of multi-soliton configurations with scattering property by using the noncommutative extension of Bäcklund transformations defined by Dai and Terng in a recent paper.

Key words: noncommutative, star product, Moyal-Weyl map, Bäcklund transformation

2000 MR subject classification: 46L87, 35Q51

Document code: A

Article ID: 1674-5647(2009)04-0349-12

1 Introduction

The study of noncommutative field theory has been carried for a few years. The original motivation derived from the discovery that a certain corner of string moduli space is described by noncommutative gauge theory (see [1]). For reviews on this subject and its connection with string and brane theory see [2, 3]. It was shown in [4] that open $N = 2$ strings in a constant B-field background induce on the world volume of n coincident D_2-branes a noncommutative generalization of a modified $U(n)$ 2+1 chiral model known as the Ward model (see [5]). Later, a supersymmetric extension of this model was introduced in [6]. These models are integrable and can be formulated as the compatibility conditions of some linear equations involving a spectral parameter. Therefore a powerful solution-generating technique, the so-called dressing method (see [7]), is employed to obtain multi-soliton solutions to the noncommutative field equations.

* **Received date:** Dec. 6, 2008.
Foundation item: The NSF (10671171) of China and the NSF (BK2007073) of Jiangsu Province, China.

By making an ansatz with only first-order poles in the spectral parameter, a wide class of no-scattering configurations were constructed in [4, 8]. The explicit two-soliton configurations with non-trivial scattering property were obtained by allowing for second-order poles in the dressing ansatz (see [9, 10]).

In a recent paper [11], all soliton solutions to the Ward model are proved to be constructed by algebraic Bäcklund transformations (BTs) and an order k limiting method. In this paper, we generalize their methods to the noncommutative setup and construct multi-soliton configurations with general pole data. The supersymmetric extension of this method is given in another paper that we have been concentrated on recently.

This paper is organized as follows. We present the noncommutative model and the auxiliary linear system in Section 2. In Section 3, after briefly reviewing the dressing method with only first-order ansatz, we give the extended algebraic BTs and get the multi-soliton configurations with only simple pole data, which coincide with the ones obtained by the dressing method. We apply the generalized limiting method and BTs to construct multi-soliton configurations with general pole data in Section 4.

2 The Noncommutative Modified 2+1 Chiral Model

Classical field theory on noncommutative space-time $\mathbb{R}^{2,1}$ may be realized in a star-product formulation or in an operator formalism. The first approach is closer to the commutative field theory. It is obtained by simply deforming the ordinary product of classical fields (or their components) to the noncommutative star product

$$(f \star g)(t,x,y) = f(t,x,y)\exp\Big\{\frac{\mathrm{i}}{2}\theta(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x})\Big\}g(t,x,y), \tag{2.1}$$

where $\theta \geq 0$ is a constant real noncommutativity parameter. Note that the time coordinate remains commutative. So the $U(n)$-valued noncommutative modified 2+1 chiral model field equation is

$$\partial_x(\varPhi^{-1} \star \partial_x \varPhi) - \partial_v(\varPhi^{-1} \star \partial_u \varPhi) = 0, \tag{2.2}$$

where $u = \frac{1}{2}(t+y)$, $v = \frac{1}{2}(t-y)$.

The operator formalism trades the star product for operator-valued spatial coordinates $(\hat{x},\ \hat{y})$ or their combinations $(\hat{z},\ \hat{\bar{z}})$, subject to

$$[t,\ \hat{x}] = [t,\ \hat{y}] = 0, \qquad [\hat{x},\ \hat{y}] = \mathrm{i}\theta \quad \Longrightarrow \quad [\hat{z},\ \hat{\bar{z}}] = 2\theta. \tag{2.3}$$

The later equation suggests the introduction of annihilation and creation operators,

$$a = \frac{1}{\sqrt{2\theta}}\hat{z} \qquad \text{and} \qquad a^\dagger = \frac{1}{\sqrt{2\theta}}\hat{\bar{z}} \quad \text{with} \quad [a,\ a^\dagger] = 1, \tag{2.4}$$

which act on a harmonic-oscillator Fock space $\mathcal{H}$ with an orthonormal basis $\{|\,\ell\rangle,\ l = 0, 1, 2, \cdots\}$ such that

$$a|\,\ell\rangle = \sqrt{\ell}\,|\,\ell-1\rangle \qquad \text{and} \qquad a^\dagger|\,\ell\rangle = \sqrt{\ell+1}\,|\,\ell+1\rangle. \tag{2.5}$$

Any function $f(t,z,\bar{z})$ can be related to an operator-valued function $\hat{f}(t) \equiv F(t,a,a^\dagger)$ acting on $\mathcal{H}$, with the help of the Moyal-Weyl map

$$f(t,z,\bar{z}) \to F(t,a,a^\dagger) = \text{Weyl-ordered } f(t, \sqrt{2\theta}a, \sqrt{2\theta}a^\dagger). \tag{2.6}$$