COMMUNICATIONS IN MATHEMATICAL RESEARCH
25(4)(2009), 329–339

Hybrid Mean Value of the Hyper Cochrane Sums*

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Communicated by Liu Jian-ya

Abstract: The main purpose of this paper is to use the analytic methods to study the hybrid mean value involving the hyper Cochrane sums, and give several sharp asymptotic formulae.

Key words: hyper Cochrane sum, hyper-Kloosterman sum, hybrid mean value
2000 MR subject classification: 11N37, 11F20
Document code: A
Article ID: 1674-5647(2009)04-0329-11

1 Introduction

For any positive integers q and n and an arbitrary integer h, the general Dedekind sums are defined by

$$S(h,n,q)=\sum_{a=1}^{q}\overline{B}_n\Big(\frac{a}{q}\Big)\overline{B}_n\Big(\frac{ha}{q}\Big),$$

where

$$\overline{B}_n(x)=\begin{cases} B_n(x-[x]), & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer,}\end{cases}$$

with $B_n(x)$ being the Bernoulli polynomial. $\overline{B}_n(x)$ is the n-th Bernoulli periodic function defined on the interval $0 < x \leq 1$.

This summation is very important, since $S(h,1,q) = s(h,q)$ is the famous Dedekind sums defined as

$$s(h,q)=\sum_{a=1}^{q}\Big(\Big(\frac{a}{q}\Big)\Big)\Big(\Big(\frac{ha}{q}\Big)\Big),$$

where

$$((x))=\begin{cases} x-[x]-\dfrac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

*Received date: Dec. 1, 2008.
Foundation item: The NSF (10671155, 10871123) and PNSF (2007A06) of China.

In [1] and [2], Zhang *et al.* have given some mean value properties of $S(h, n, q)$.

In October 2000, during his visit to Xi'an, Todd Cochrane introduced the following sums analogous to the Dedekind sums as

$$C(h, q) = \sum_{a=1}^{q}{}' \Big(\Big(\frac{a}{q}\Big)\Big)\Big(\Big(\frac{h\overline{a}}{q}\Big)\Big),$$

where $\sum'$ denotes the summation over all a such that

$$(a,\ q) = 1,$$

and

$$a\overline{a} \equiv 1 \ (\mathrm{mod}\ q).$$

He advised us to study the arithmetical properties and mean value distributive properties of Cochrane sums $C(h,\ q)$. Yi and Zhang[3] studied the upper bound estimate of them, and obtained that

$$|C(h, q)| \ll q^{\frac{1}{2}} d(q) \ln^2(q),$$

where $d(q)$ is the divisor function. Similarly, Xu and Zhang[4] defined the high-dimensional Cochrane sums as

$$C(h, k, q) = \sum_{a_1=1}^{q}{}' \sum_{a_2=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' \Big(\Big(\frac{a_1}{q}\Big)\Big)\Big(\Big(\frac{a_2}{q}\Big)\Big) \cdots \Big(\Big(\frac{a_k}{q}\Big)\Big)\Big(\Big(\frac{h\overline{a_1 a_2 \cdots a_k}}{q}\Big)\Big),$$

and obtained

$$|C(h, k, q)| \ll \frac{2^{(k+1)^2}}{\pi^{k+1}} q^{\frac{k}{2}} d(q) (2^{k+2}k)^{\omega(q)} \ln^{k+1}(q),$$

where $\omega(q)$ denotes the number of all different prime divisors of q. Soon after that, Liu[5] improved the upper bound with a simple method.

Moreover, Zhang[6] found that there exist some interesting connections between $C(h, q)$ and Kloosterman sums

$$K(m, n;\ q) = \sum_{b=1}^{q}{}' \mathrm{e}\Big(\frac{mb + n\overline{b}}{q}\Big),$$

where

$$\mathrm{e}(y) = \mathrm{e}^{2y\mathrm{i}\pi}.$$

For example, if q is a square-full number (i.e., $p|q$ if and only if $p^2|q$), then we have the following asymptotic formula:

$$\sum_{h=1}^{q}{}' K(h, 1;\ q) C(h, q) = \frac{-1}{2\pi^2} q\phi(q) + O\Big(q \exp\Big\{\frac{3 \ln q}{\ln \ln q}\Big\}\Big).$$

For general integer $q \geq 3$, Zhang[7] proved the asymptotic formula

$$\sum_{h=1}^{q}{}' K(h, 1;\ q) C(h, q) = \frac{-1}{2\pi^2} q\phi(q) \prod_{p \| q} \Big(1 - \frac{1}{p(p-1)}\Big) + O\Big(q^{\frac{3}{2}+\epsilon}\Big),$$

where ϵ is any fixed positive number. For the r-th Kloosterman sums which are defined as

$$K(m, n, r;\ q) = \sum_{b=1}^{q}{}' \mathrm{e}\left(\frac{mb^r + n\overline{b^r}}{q}\right),$$