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Lipschitz Estimates for Commutators of N-dimensional Fractional Hardy Operators*

ZHENG QING-YU AND FU ZUN-WEI
(*Department of Mathematics, Linyi Normal University, Linyi, Shandong, 276005*)

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Abstract: In this paper, it is proved that the commutator $\mathcal{H}_{\beta,b}$ which is generated by the n-dimensional fractional Hardy operator $\mathcal{H}_\beta$ and $b \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $0 < \alpha < 1$, $1 < p, q < \infty$ and $1/p - 1/q = (\alpha + \beta)/n$. Furthermore, the boundedness of $\mathcal{H}_{\beta,b}$ on the homogenous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is obtained.

Key words: commutator, n-dimensional fractional Hardy operator, Lipschitz function, Herz space

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1 Introduction

Let f be a non-negative integrable function on $\mathbb{R}^+$. The classical Hardy operators are defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)\mathrm{d}t, \quad H^*f(x) := \int_x^\infty \frac{f(t)}{t}\mathrm{d}t, \qquad x > 0.$$

The Hardy operators H and H^* are adjoint mutually, that is

$$\int_0^\infty g(x)Hf(x)\mathrm{d}x = \int_0^\infty f(x)H^*g(x)\mathrm{d}x, \tag{1.1}$$

where $f \in L^p(\mathbb{R}^+)$, $g \in L^q(\mathbb{R}^+)$, $1 < p < \infty$ and $1/p + 1/q = 1$.

The most celebrated Hardy's integral inequality established in [1] by Hardy which can be stated as follows.

Theorem 1.1 *If $1 < p < \infty$, then*

$$\|Hf\|_{L^p(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}, \qquad \|H^*f\|_{L^q(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^q(\mathbb{R}^+)}. \tag{1.2}$$

Moreover,

$$\|H\|_{L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)} = \|H^*\|_{L^q(\mathbb{R}^+)\to L^q(\mathbb{R}^+)} = \frac{p}{p-1}, \tag{1.3}$$

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where $1/p + 1/q = 1$.

The inequality (1.2) is a by-product of Hilbert's double series theorem. The Hardy's integral inequalities have drawn considerable attention (see [2]). In 1995, Christ and Grafakos[3] obtained that

Theorem 1.2 *Let f be a locally integrable function on $\mathbb{R}^n$, $1 < p < \infty$. Then*

$$\|\mathcal{H}f\|_{L^p(\mathbb{R}^n)} \leq \frac{p\nu_n}{p-1}\|f\|_{L^p(\mathbb{R}^n)}, \tag{1.4}$$

where $\mathcal{H}$ is the n-dimensional Hardy operator which is defined by

$$\mathcal{H}f(x) := \frac{1}{|x|^n}\int_{|t|<|x|} f(t)\mathrm{d}t, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

and the constant $p\nu_n/(p-1)$ is the best possible, $\nu_n = \pi^{n/2}/\Gamma(1+n/2)$.

In [4], Fu *et al.* defined the n-dimensional fractional Hardy operators as follows.

Definition 1.1 *Let f be a locally integrable function on $\mathbb{R}^n$, $\beta \in \mathbb{R}^1$. The n-dimensional fractional Hardy operators are defined by*

$$\mathcal{H}_\beta f(x) := \frac{1}{|x|^{n-\beta}}\int_{|t|<|x|} f(t)\mathrm{d}t, \quad \mathcal{H}^*_\beta f(x) := \int_{|t|\geq|x|} \frac{f(t)}{|t|^{n-\beta}}\mathrm{d}t, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

Obviously, when $\beta = 0$, $\mathcal{H}_\beta$ is just $\mathcal{H}$. $\mathcal{H}_\beta$ and $\mathcal{H}^*_\beta$ satisfy

$$\int_{\mathbb{R}^n} g(x)\mathcal{H}_\beta f(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\mathcal{H}^*_\beta g(x)\mathrm{d}x. \tag{1.5}$$

Remark 1.1 It should be pointed out the range of β is $[0,\ n)$ in [4].

Definition 1.2[4] *Let b be a locally integrable function on $\mathbb{R}^n$, $\beta \in \mathbb{R}^1$. We define the commutators of n-dimensional fractional Hardy operators as follows:*

$$\mathcal{H}_{\beta,b} f := b\mathcal{H}_\beta f - \mathcal{H}_\beta(fb), \qquad \mathcal{H}^*_{\beta,b} f := b\mathcal{H}^*_\beta f - \mathcal{H}^*_\beta(fb).$$

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_E is the characteristic function of set E.

Definition 1.3[5] *Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q < \infty$. The homogenous Herz space $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha,p}_q(\mathbb{R}^n) := \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \right\}^{1/p}.$$

Remark 1.2 $\dot{K}^{0,p}_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\dot{K}^{\alpha/p,p}_p(\mathbb{R}^n) = L^p(|x|^\alpha dx)$ for all $0 < p \leq \infty$ and $\alpha \in \mathbb{R}$.

In [4], the authors have obtained that the commutator $\mathcal{H}_{\beta,b}$ which is generated by n-dimensional fractional Hardy operator and a locally integrable function b is a bounded