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A Linearized Spectral-Galerkin Method for Three-Dimensional Riesz-Like Space Fractional Nonlinear Coupled Reaction-Diffusion Equations

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Abstract. In this paper, we establish a novel fractional model arising in the chemical reaction and develop an efficient spectral method for the three-dimensional Riesz-like space fractional nonlinear coupled reaction-diffusion equations. Based on the backward difference method for time stepping and the Legendre-Galerkin spectral method for space discretization, we construct a fully discrete numerical scheme which leads to a linear algebraic system. Then a direct method based on the matrix diagonalization approach is proposed to solve the linear algebraic system, where the cost of the algorithm is of a small multiple of N^4 (N is the polynomial degree in each spatial coordinate) flops for each time level. In addition, the stability and convergence analysis are rigorously established. We obtain the optimal error estimate in space, and the results also show that the fully discrete scheme is unconditionally stable and convergent of order one in time. Furthermore, numerical experiments are presented to confirm the theoretical claims. As the applications of the proposed method, the fractional Gray-Scott model is solved to capture the pattern formation with an analysis of the properties of the fractional powers.

AMS subject classifications: 26A33, 65M06, 65M12, 65M15, 35R11

Key words: Riesz-like fractional derivative, nonlinear coupled reaction-diffusion equations, Legendre-Galerkin spectral method, stability and convergence.

1. Introduction

In the last few decades, increasing studies [2,3] have revealed the fact that one particle's long walk can have a long correlation length [52]. Such kind of long-range corre-

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lation and large jumps can lead to the significant deviations from the stochastic process of Brownian motion for the classical diffusion process, which indicates that the Lévy process is more appropriate. To model the transport processes which exhibit anomalous diffusion, recent investigations [33, 43] have shown that the fractional diffusion equations can provide a powerful and accurate description, however the second-order diffusion equations cannot properly model such processes. Consequently, fractional diffusion equations have gained considerable attention and popularity, and these kinds of models have been widely applied in soil contamination, chemical processes, underground water flow, etc (see [27, 28, 53]).

The increasing applications of the fractional models to the realistic problems open up a natural question: How to solve the fractional differential equations? For most of these equations, we cannot obtain their exact analytic solutions [39]. Moreover, compared to the integer-order differential equations, the fractional ones encounter the mathematical difficulties:

(i) The fractional differential operators are expressed in the integral forms (nonlocal property).

(ii) The adjoint of a fractional differential operator is not the negative of itself [52].

Therefore, developing efficient schemes for the numerical solutions of fractional differential equations has become an important and challenging task. Thanks to the efforts by many authors, a number of numerical methods have been developed to solve fractional differential equations, such as finite difference methods [8, 24, 51, 55, 56], finite element methods [11–14, 21, 25, 42], and spectral methods [9, 18, 26, 29, 30, 34]. We find that:

(i) The existing literature mainly investigated the numerical solutions of the single fractional differential equation. Up to now, there have existed limited works for numerical approximations of the coupled fractional differential equations, see, for example, [36, 37, 50]. In fact, the systems of the coupled fractional reaction-diffusion equations play a significant role in the study of self-organization phenomena in various areas of science and engineering, e.g., spatially inhomogeneous media [19] and intracellular dynamics [35].

(ii) Until now, most of these numerical methods have focused on the linear fractional models [54]. Relatively, only a few works have been carried out to solve the nonlinear fractional differential equations because the appearance of the nonlinear term leads to many difficulties in the progress of the numerical methods [18].

(iii) Most recent studies have been absorbed in the numerical solutions of the fractional differential equations only in one and two space dimensions. Although the fractional models for three-dimensional cases are much more useful in the real applications, the existing works on the numerical solutions of such problems are quite sparse.