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The Sufficient and Necessary Condition of Lagrange Stability of Quasi-periodic Pendulum Type Equations*

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Abstract: The quasi-periodic pendulum type equations are considered. A sufficient and necessary condition of Lagrange stability for this kind of equations is obtained. The result obtained answers a problem proposed by Moser under the quasi-periodic case.

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1 Introduction

The Lagrange stability of pendulum type equations is an important topic, which is proposed by Moser[1]. Moser[2], Levi[3] and You[4] investigated such topic for the periodic situation, respectively. In particular, You obtained a sufficient and necessary condition for Lagrange stability of the equation (1.1) in [4].

Recently, Bibikov[5] developed a KAM theorem for nearly integrable Hamiltonian systems with one degree of freedom under the quasi-periodic perturbation. In fact, his KAM theorem is of parameter type. Using this theorem he discussed the stability of equilibrium of a class of the second order nonlinear differential equations.

In this note we study quasi-periodic pendulum type equations. Under the standard Diophantine condition of frequency ω, a sufficient and necessary condition of Lagrange stability for quasi-periodic pendulum type equations is obtained. This answers Moser's problem under the quasi-periodic case.

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We consider a nonlinear pendulum type equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + p(t, x) = 0, \tag{1.1}$$

where

$$p(t, x + 1) = p(t, x),$$

and $p(t, x)$ is a quasi-periodic function in t with basic frequencies $\omega = (\omega_1, \cdots, \omega_n)$, that is,

$$p(t, x) = f(\omega t, x) \tag{1.2}$$

for some function $f(\theta, x)$ defined on $T^n \times T^1$. Here $T^n = R^n/Z^n$ is an n-dimensional torus.

Assume that $f(\theta, x)$ is a real analytic function on $T^n \times T$ and the frequency ω satisfies Diophantine condition as follows:

$$|\langle k, \omega\rangle| \geq \gamma |k|^{-(n+1)}, \qquad 0 \neq k \in Z^n \tag{1.3}$$

for a given $\gamma > 0$, where $\langle \cdot\,, \cdot \rangle$ denotes the usual inner product.

We are in a position to state the main result of this paper.

Theorem 1.1 *Assume that* (1.3) *holds. Then system* (1.1) *is Lagrange stable if and only if*

$$\int_{T^n \times T^1} f(\theta, x) \mathrm{d}\theta \mathrm{d}x = 0. \tag{1.4}$$

Moreover, if (1.3) *and* (1.4) *hold, equation* (1.1) *possesses infinitely many quasi-periodic solutions with* $n + 1$ *basic frequencies* (*including* $\omega_1, \cdots, \omega_n$).

- Diophantine condition (1.3) can be replaced by a general form

$$|\langle k, \omega\rangle| \geq \gamma |k|^{-\tau_*}, \qquad 0 \neq k \in Z^n \tag{1.5}$$

  with some constant $\tau_* > n$. Here we assume (1.3) for the convenience of the proof of Theorem 1.1.

- Huang[6] considered a class of almost periodic pendulum-type equations. He proved the existence of unbounded solutions of the equations. Summing up the works developed by Mose[2], Levi[3], You[4] and Huang[6], respectively, and Theorem 1.1, we can obtain a satisfactory answer to Moser's problem.

- Recently, Lin and Wang[7] have concerned with a dual quasi-periodic system as follows:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\partial g}{\partial x}(t, x) = 0, \tag{1.6}$$

  where $g(t, x)$ is quasi-periodic in t and x with frequencies $\Omega^1 = (\omega_1, \cdots, \omega_n)$ and $\Omega^2 = (\omega_{n+1}, \cdots, \omega_{n+m})$, respectively. Under the assumptions

$$(\Omega^1, \Omega^2) \in O_\gamma = \big\{(\Omega^1, \Omega^2) \in R^{n+m} : |\langle k, \Omega^1\rangle + \langle l, \Omega^2\rangle| \geq \gamma(|k| + |l|)^{-\tau_*}, \\ \forall\ 0 \neq (k, l) \in Z^{n+m},\ \tau_* > n + m\big\}$$

  and

$$\forall\ j \in N,\ \exists\ A(j) \geq j, \qquad \text{s.t. } (\Omega^1, A(j)\Omega^2) \in O_\gamma,$$

  they proved that all the solutions of (1.6) are bounded (see [7]). It is easy to find that as $m = 1$, their modified Diophantine condition is stronger than (1.5); in addition, the result of [7] is a sufficient condition to ensure Lagrange stability. This differs from Theorem 1.1.