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A Riesz Product Type Measure on the Cantor Group*

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Abstract: A Riesz type product as

$$P_n = \prod_{j=1}^{n}(1 + a\omega_j + b\omega_{j+1})$$

is studied, where a, b are two real numbers with $|a| + |b| < 1$, and $\{\omega_j\}$ are independent random variables taking values in $\{-1, 1\}$ with equal probability. Let $\mathrm{d}\omega$ be the normalized Haar measure on the Cantor group $\Omega = \{-1, 1\}^N$. The sequence of probability measures $\left\{\dfrac{P_n \mathrm{d}\omega}{E(P_n)}\right\}$ is showed to converge weakly to a unique continuous measure on Ω, and the obtained measure is singular with respect to $\mathrm{d}\omega$.

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1 Introduction

The Riesz product is a kind of lacunary series of trigonometric. It is an important topic in the field of harmonic analysis. The classical Riesz product measure is first introduced on the circle group $T = R/Z$ by Riesz, and later generalized by Zygmund[1] as the weak limit of finite Riesz products

$$\prod_{1}^{N}(1 + a_n \cos(2\pi\lambda_n t))$$

as N tends to infinity, where a_n's are bounded by 1 and the integers λ_n's are lacunary in the sense $\lambda_{n+1}/\lambda_n \geq 3$. In other words, there is a Radon measure μ such that

$$\lim_{N\to\infty} \int_T f(t) \prod_{1}^{N}(1 + a_n \cos(2\pi\lambda_n t))\mathrm{d}t = \int_T f(t)\mathrm{d}\mu(t), \qquad \forall f \in C(T).$$

Moreover, this measure is continuous, that is,

$$\mu(\{t\}) = 0, \qquad \forall t \in T.$$

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Later, Hewitt and Zuckerman[2] defined Riesz products on a general non-discrete compact abelian group. A short description of their approach is as follows.

Let G be a nondiscrete compact abelian group with discrete dual group Γ, Λ be a subset of Γ, and $W(\Lambda)$ be the set of all elements $\gamma \in \Gamma$ in the form of

$$\gamma = \lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \cdots \lambda_n^{\epsilon_n}, \tag{1.1}$$

where $\epsilon_k \in \{-1, 1\}$ and λ_k are distinct elements of Λ. Suppose that Λ satisfies the requirement that each element of $W(\Lambda)$ has a unique representation of the form (1.1) up to the order of the factors, and let α be any complex function on Λ bounded by 1. For any finite set $\Phi \subset \Lambda$, define a Riesz product on G as follows:

$$P(\Phi, \alpha) = \prod \{1 + \mathrm{Re}[\alpha(\lambda)\lambda] : \lambda \in \Phi\}.$$

Hewitt and Zuckerman[2] showed that there exists a unique continuous probability measure $\mu_{\alpha,\lambda}$ on G which is the weak limit of $P(\Phi, \alpha)\mathrm{d}m$ in the topology of $M(G)$, where $M(G)$ is the convolution algebra of all Radon measures on G and m is the normalized Haar measure on G. A famous theorem of Kakutani[3] says that $\mu_{\alpha,\lambda}$ is either absolutely continuous or singular with respect to the Lebesgue-Haar measure on G, according to whether $\alpha \in l^2(\Lambda)$ or not.

The Riesz product is proved to be a source of powerful idea that can be used to produce concrete examples of measures with desired properties, such as singularity and multifractal structure. For the latter topic, refer to Peyriere[4] and Fan[5].

In this paper, we study a Riesz product type measure on the Cantor group. Throughout this paper, let

$$\Omega = \prod_1^{\infty} \Omega_j = \{-1, 1\}^N$$

be the cartesian product with all factors equal to

$$\Omega_j = \{-1, 1\}, \qquad \forall j \geq 1,$$

and write its elements

$$\varepsilon = (\varepsilon_n)_{n \in \mathbf{N}}$$

or

$$\varepsilon = \varepsilon_1 \varepsilon_2 \cdots$$

Ω is well known as an abelian group under the operation of pointwise product. With the discrete topology on each factor, the product topology on Ω makes it a compact abelian group, the so-called Cantor group. This topology can also be induced by a metric that the distance between two elements $\varepsilon = (\varepsilon_n)_{n \in \mathbf{N}}$, $\delta = (\delta_n)_{n \in \mathbf{N}}$ in Ω equals to

$$2^{-\inf\{n : \varepsilon_i = \delta_i, 0 \leq i \leq n, \varepsilon_{n+1} \neq \delta_{n+1}\}}.$$

Denote the projection $\omega_n : \Omega \to \{-1, 1\}$ by

$$\omega_n(\varepsilon) = \varepsilon_n.$$

Elements in the dual group Γ of Ω, which are continuous group homomorphisms from Ω into the multiplicative group of complex numbers of modulus 1, are provided by the projection functions. Precisely, let

$$\mathcal{R} = \{\omega_n : n \in \mathbf{N}\} \subset \Gamma.$$