Communications
in
Mathematical
Research
26(2)(2010), 119–130

Computation of the Rational Representation for Solutions of High-dimensional Systems*

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Communicated by Ma Fu-ming

Abstract: This paper deals with the representation of the solutions of a polynomial system, and concentrates on the high-dimensional case. Based on the rational univariate representation of zero-dimensional polynomial systems, we give a new description called rational representation for the solutions of a high-dimensional polynomial system and propose an algorithm for computing it. By this way all the solutions of any high-dimensional polynomial system can be represented by a set of so-called rational-representation sets.

Key words: rational univariate representation, high-dimensional ideal, maximally independent set, rational representation, irreducible component

2000 MR subject classification: 65H10, 14Q15, 13P10

Document code: A

Article ID: 1674-5647(2010)02-0119-12

1 Introduction

If an algebraic system has infinitely many zeros, that is, the variety defined by the algebraic system is high-dimensional, how to describe its zero-set is significant but difficult. Wu[1] proposed a method for computing the irreducible components of the variety by means of characteristic set method, or Wu's method. Some others studied the primary decomposition of the ideal defined by the algebraic system with Gröbner[2] bases. Feng and Zhang[3] discussed the eigenvalue method for solving high-dimensional algebraic systems. Éric studied the representation of the solutions of a polynomial system by triangular sets (see [4] and [5]).

In order to represent the solution in such a way as to allow any arithmetical operations over the arithmetical expressions of its coordinates, based on the rational univariate representation (RUR) for solving zero-dimensional systems (see [6]), we give the so-called rational representation for describing all the solutions of a high-dimensional polynomial system. We

***Received date:** Nov. 13, 2008.
Foundation item: The National Grand Fundamental Research 973 Program (2004CB318000) of China.

reduce to dimension zero by placing the independent variables in the base field, so the solutions can be represented by the rational univariate representation with coefficients in a rational function field. Using this method we can get a so-called rational-representation set. The solutions represented by the rational-representation set are defined as: the independent variables at a point which is not a root of some polynomial in the rational-representation set are specialized, and then the other coordinates of the corresponding solution can be expressed as rational functions at the zeros of an univariate polynomial. We show that all the solutions of any polynomial system with positive dimension can be represented by a set of rational-representation sets. A special feature of our rational representation is that it is for all the solutions.

2 RUR for Zero-dimensional Ideals

In this section, let K be a field of characteristic zero and L an algebraically closed field containing K. Let $K[X_1, \cdots, X_n]$ be the polynomial ring over K, and I a zero-dimensional ideal of $K[X_1, \cdots, X_n]$. We set

$$A_K(I) = K[X_1, \cdots, X_n]/I,$$

and denote by $Z_L(I) \subset L^n$ the zero-set of I in L^n.

Definition 2.1 [7] *A polynomial $u \in K[X_1, \cdots, X_n]$ separates $Z_L(I)$, if*

$$\forall\ \alpha, \beta \in Z_L(I), \qquad \alpha \neq \beta \Rightarrow u(\alpha) \neq u(\beta).$$

Definition 2.2 [6] *Let $I \subset K[X_1, \cdots, X_n]$ be a zero-dimensional ideal. For all $h \in K[X_1, \cdots, X_n]$, we denote by m_h^I the K-linear map:*

$$m_h^I : A_K(I) \to A_K(I)$$
$$\overline{q} \mapsto \overline{hq},$$

where $\overline{q}$ denotes the residue class in $A_K(I)$ of any polynomial $q \in K[X_1, \cdots, X_n]$.

Definition 2.3 [6] *Let $I \subset K[X_1, \cdots, X_n]$ be a zero-dimensional ideal, u any element in $K[X_1, \cdots, X_n]$ and $\mathscr{X}_u$ the characteristic polynomial of m_u^I. For any $h \in K[X_1, \cdots, X_n]$, we define*

$$g_u(h, T) = \sum_{\alpha \in Z_L(I)} \mu(\alpha) h(\alpha) \prod_{y \neq u(\alpha), y \in Z_L(\mathscr{X}_u)} (T - y),$$

where $\mu(\alpha)$ denotes the multiplicity of α. If u separates $Z_L(I)$, the $(n+2)$-tuple

$$\{\mathscr{X}_u, g_u(1, T), g_u(X_1, T), \cdots, g_u(X_n, T)\}$$

is called the rational univariate representation (RUR) of I associated to u.

Theorem 2.1 [6] *The rational univariate representation associated to a separating element u satisfies the following properties:*

(1) *The polynomials $\mathscr{X}_u$, $g_u(1, T)$, $g_u(X_1, T)$, $\cdots$, $g_u(X_n, T)$ are elements of $K[T]$;*

(2) *The roots of $\mathscr{X}_u(T)$ in L are exactly the scalars $u(\alpha)$, $\alpha \in Z_L(I)$;*

(3) *For each $\alpha \in Z_L(I)$, $\alpha = \Big(\dfrac{g_u(X_1, u(\alpha))}{g_u(1, u(\alpha))}, \cdots, \dfrac{g_u(X_n, u(\alpha))}{g_u(1, u(\alpha))}\Big)$ holds.*