Communications
in
Mathematical
Research
26(3)(2010), 211–218

Convergence Rate of Empirical Bayes for Two-parameter Exponential Distribution with Replicated Data*

Li Nai-yi
(*Faculty of Science, Guangdong Ocean University, Zhanjiang, Guangdong, 524088*)

Communicated by Wang De-hui

Abstract: In this paper, empirical Bayes test for a parameter θ of two-parameter exponential distribution is investigated with replicated past data. Under some conditions, the asymptotically optimal property is obtained. It is indicated that the rate of convergence can be very close to $O(N^{-\frac{1}{2}})$ in this case that a parameter μ is known.

Key words: replicated data, empirical Bayes, asymptotic optimality
2000 MR subject classification: 62G07
Document code: A
Article ID: 1674-5647(2010)03-0211-08

1 Introduction

Empirical Bayes test has been applied in many distributions (see [1–5]). Recently, in more experiments, past samples are independently observed with repetition. Then, via empirical Bayes approach, we test the parameter in two-parameter exponential distribution with replicated past data.

Let X have conditional density of the form

$$f(x|\theta) = (\theta + \mu x) \exp\Big\{ -\theta x - \frac{1}{2}\mu x^2 \Big\}, \tag{1.1}$$

where μ is a known positive constant, and θ is an unknown parameter.

The hypothesis to be tested is

$$\mathrm{H}_0: \ \theta_1 \le \theta \le \theta_2 \quad \text{against} \quad \mathrm{H}_1: \ \theta < \theta_1 \text{ or } \theta > \theta_2, \tag{1.2}$$

where θ_1 and θ_2 are given constants. Define $\theta_0 = \dfrac{\theta_1 + \theta_2}{2}$ and $\theta_3 = \dfrac{\theta_2 - \theta_1}{2}$. Then the test problem of (1.2) is equivalent to

$$\mathrm{H}_0^*: \ |\theta - \theta_0| \le \theta_3 \quad \text{against} \quad \mathrm{H}_1^*: \ |\theta - \theta_0| > \theta_3. \tag{1.3}$$

*__Received date:__ Sept. 5, 2008.
Foundation item: The NSF (10661003) of China and the NSF (1012138, 0612163) of Guangdong Ocean University.

Let the loss function be

$$L_i(\theta, d_i) = (1-i)a[(\theta-\theta_0)^2 - \theta_3^2]I_{[|\theta-\theta_0|>\theta_3]} + ia[\theta_3^2 - (\theta-\theta_0)^2]I_{[|\theta-\theta_0|\le\theta_3]},$$

where $i = 0, 1$, $a > 0$, $d = \{d_0,\ d_1\}$ is the decision space, d_0 and d_1 indicate acceptance and rejection of H_0^*.

Note that

$$\Omega = \{x|x > 0\}$$

is the sample space, and

$$\Theta\Big\{\theta > 0\Big|\int_\Omega f(x|\theta)\mathrm{d}x = 1\Big\}$$

is the parameter space. Suppose that the parameter θ has a unknown prior distribution $G(\theta)$. We obtain the randomized decision function

$$\delta(x) = P(\text{accepting } \mathrm{H}_0^* \mid X = x). \tag{1.4}$$

Then, the Bayes risk function of $\delta(x)$ is shown by

$$\begin{aligned} R(\delta(x), G(\theta)) &= \int_\Theta\int_\Omega [L_0(\theta, d_0)f(x|\theta)\delta(x) + L_1(\theta, d_1)f(x|\theta)(1-\delta(x))]\mathrm{d}x\mathrm{d}G(\theta) \\ &= a\int_\Omega \beta(x)\delta(x)\mathrm{d}x + C_G, \end{aligned} \tag{1.5}$$

where

$$C_G = \int_\Theta L_1(\theta, d_1)\mathrm{d}G(\theta), \qquad \beta(x) = \int_\Theta [(\theta-\theta_0)^2 - \theta_3^2]f(x|\theta)\mathrm{d}G(\theta). \tag{1.6}$$

The marginal density of X is

$$\begin{aligned} f_G(x) &= \int_\Theta f(x|\theta)\mathrm{d}G(\theta) \\ &= \int_\Theta (\theta + \mu x)\exp\Big\{-\theta x - \frac{1}{2}\mu x^2\Big\}\mathrm{d}G(\theta). \end{aligned}$$

Let

$$v_G(x) = \int_\Theta \exp\Big\{-\theta x - \frac{1}{2}\mu x^2\Big\}\mathrm{d}G(\theta).$$

Hence

$$\begin{aligned} v_G^{(1)}(x) &= -\int_\Theta (\theta + \mu x)\exp\Big\{-\theta x - \frac{1}{2}\mu x^2\Big\}\mathrm{d}G(\theta) \\ &= -f_G(x), \end{aligned}$$

where $v_G^{(1)}(x)$ is the derivative of $v_G(x)$. Then

$$\int_x^\infty f_G(x)\mathrm{d}x = v_G(x). \tag{1.7}$$

By (1.6) and simple calculation, we have

$$\beta(x) = f_G^{(2)}(x) + L(x)f_G^{(1)}(x) - \mu L(x)v_G(x) + W(x)f_G(x), \tag{1.8}$$

where

$$L(x) = 2\mu x + 2\theta_0, \qquad W(x) = \mu^2x^2 + 2\mu\theta_0 x + 3\mu + \theta_0^2 - \theta_3^2,$$

and $f_G^{(1)}(x)$, $f_G^{(2)}(x)$ are first and second order derivatives of $f_G(x)$, respectively.

By using (1.5), Bayes test function is obtained as follows

$$\delta_G(x) = \begin{cases} 1, & \beta(x) \le 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.9}$$