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A Note on Upper Convex Density*

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Abstract: For a self-similar set E satisfying the open set condition, upper convex density is an important concept for the computation of its Hausdorff measure, and it is well known that the set of relative interior points with upper convex density 1 has a full Hausdorff measure. But whether the upper convex densities of E at all the relative interior points are equal to 1? In other words, whether there exists a relative interior point of E such that the upper convex density of E at this point is less than 1? In this paper, the authors construct a self-similar set satisfying the open set condition, which has a relative interior point with upper convex density less than 1. Thereby, the above problem is sufficiently answered.

Key words: self-similar set, open set condition, upper convex density

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1 Introduction

It is well known that the theory of Hausdorff measure and Hausdorf dimension is the basis of fractal geometry, so how to compute or estimate the Hausdorff measures and Hausdorff dimensions of the fractal sets is an important problem. In general, to compute or estimate the Hausdorff measures and Hausdorff dimensions of fractals is very difficult and to compute the Hausdorff measures of fractals is more difficult. Up to now, the fractals studied more successfully are the self-similar sets satisfying the open set condition. But up to now, just for such a simplest class of fractals, to compute their Hausdorff measures is still very difficult. Especially for the fractals with Hausdorff dimension larger than 1, how to compute their Hausdorff measures remains an open problem.

Upper convex density is an important concept which plays a key role in the computation of Hausdorff measure. There are two limit processes in its definition, so it is also difficult to calculate or estimate the exact value of upper convex density of a point in a self-similar set by its definition. Under a certain sense, the problem of computation of Hausdorff measure

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and that of computation of the upper convex density of a point in a self-similar set are equivalent.

In this paper, we always assume that $\dim_H$ denotes the Hausdorff dimension, $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure and $|\cdot|$ denotes diameter.

For a self-similar set E satisfying the open set condition, we know that the set of relative interior points (see the following introduction) of E has a full Hausdorff measure, i.e.,

$$\mathcal{H}^s(E) = \mathcal{H}^s(\mathrm{int}(E)),$$

where $\mathrm{int}(E)$ denotes the set of the relative interior points of E. So the upper convex densities of E at almost every relative interior points are 1. Naturally, a problem arises as in the following problem A.

Problem A Whether the upper convex densities of E at all the relative interior points are 1? In other words, whether there exists a relative interior point of E such that the upper convex density of E at this point is less than 1?

In the present paper, a self-similar set with the open set condition is constructed, which has a relative interior point with upper convex density less than 1, so the above problem A is sufficiently answered.

Some definitions, notations and known results are taken from references [1]–[3].

Let $D \subset \mathbf{R}^n$ be a closed set. $S : D \to D$ is called contractive if there is c $(0 < c < 1)$ such that

$$d(S(x),\ S(y)) \leqslant cd(x, y), \qquad \forall x, y \in D. \tag{1.1}$$

We say that S is a contracting similarity if

$$d(S(x),\ S(y)) = cd(x,\ y), \tag{1.2}$$

where c is called the similarity ratio of S. Obviously, any contracting similarity is contractive.

Let $S_i : D \to D$ be contracting similarities with the similar ratios c_i $(i = 1, 2 \cdots, m,$ $m \geqslant 2)$. We say that $\{S_i\}$ satisfies the open set condition (OSC) if there exists a nonempty bounded open set V such that

$$\bigcup_{i=1}^{m} S_i(V) \subset V, \qquad S_i(V) \bigcap S_j(V) = \emptyset, \quad i \neq j,\ 1 \leqslant i, j \leqslant m. \tag{1.3}$$

Hutchinson[4] obtained the following result.

Proposition 1.1[4] *Let $S_i : D \to D$ be contracting similarities with the similar ratios c_i $(i = 1, 2 \cdots, m,\ m \geqslant 2)$. Then there exists a unique nonempty compact set $E \subset \mathbf{R}^n$ satisfying*

$$E = \bigcup_{i=1}^{m} S_i(E). \tag{1.4}$$

The set E is called a self-similar set for the iterated function system (IFS) $\{S_1, \cdots, S_m\}$. Here we always assume that E satisfies the OSC. It is easy to prove $E \subset \overline{V}$ (V is the open set in the OSC). Furthermore, if

$$S_i(E) \bigcap S_j(E) = \emptyset, \qquad 0 < i < j \leqslant m, \tag{1.5}$$