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Quadratic Lyapunov Function and Exponential Dichotomy on Time Scales*

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Abstract: In this paper, we study the relationship between exponential dichotomy and quadratic Lyapunov function for the linear equation $x^\Delta = A(t)x$ on time scales. Moreover, for the nonlinear perturbed equation $x^\Delta = A(t)x + f(t, x)$ we give the instability of the zero solution when f is sufficiently small.

Key words: quadratic Lyapunov function, exponential dichotomy, time scale, instability

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1 Introduction

Exponential dichotomy plays an important role in the theory of nonautonomous dynamical systems. When people deal with the nonlinear problems which are the perturbation of the linear ones, exponential dichotomy is very useful. Copple[1] studied in detail the exponential dichotomy for ordinary differential equations. Coff and Schäffer[2] studied exponential dichotomy for difference equations. In 1990, Hilger[3] introduced the theory of time scales, and then there are numerous works using this notion to unify and generalize theories of continuous and discrete dynamical systems (see [3]–[5]).

Recently, Barreia and Valls[6] studied the relationship between nonuniform exponential dichotomy and quadratic Lyapunov function. In this paper, we firstly introduce strict quadratic Lyapunov function on time scales. Then we study the relationship between exponential dichotomy and strict quadratic Lyapunov function on time scales. We obtain that the linear equation $x^\Delta = A(t)x$ has a strict quadratic Lyapunov function if it admits strong exponential dichotomy; conversely, the linear equation admits exponential dichotomy if it has a strict quadratic Lyapunov function with some property. And by quadratic Lyapunov function, we investigate the instability of the zero solution of the nonlinear perturbed equa-

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tion. The stability and instability of solutions to the nonlinear perturbed equation on time scales have been studied in [7] and [8] by considering appropriate eigenvalue conditions, and in [9] by assuming the existence of Lyapunov functions.

This paper is organized as follows. In next section, we review some useful notions and basic properties on time scales. In Section 3, the main concepts, exponential dichotomy and strict quadratic Lyapunov function on time scales, are given. Furthermore, our main results are stated and proved. Finally, we study instability of the zero solution to the nonlinear perturbed equation on time scales in Section 4.

2 Preliminaries on Time Scales

For the convenience of readers, we review some preliminary definitions and theories on time scales. The reader may refer to [5] for details. Let $\mathbf{T}$ be a time scale which is an arbitrary nonempty closed subset of the real numbers. Such as the sets of real numbers and integers are the special time scales and the union of arbitrary nonempty closed intervals is also a time scale.

Definition 2.1 *Let $\mathbf{T}$ be a time scale. The forward jump operator is defined by*

$$\sigma(t) := \inf\{s \in \mathbf{T} : s > t\}$$

for every $t \in \mathbf{T}$. Let $\mu(t) := \sigma(t) - t$ be the graininess function.

It is clear that the graininess function $\mu(t)$ is nonnegative. In this paper, we always suppose that the graininess function $\mu(t)$ is bounded.

Definition 2.2 *A function $f : \mathbf{T} \to \mathbf{R}^n$ is called rd-continuous if it is continuous at right dense points in $\mathbf{T}$ and left-sided limits exist at left dense points in $\mathbf{T}$.*

We denote the set of rd-continuous functions by $C_{rd}(\mathbf{T}, \mathbf{R}^n)$.

Definition 2.3 *A function $f : \mathbf{T} \to \mathbf{R}^n$ is called differentiable at $t \in \mathbf{T}$, if for any $\varepsilon > 0$, there exists a $\mathbf{T}$-neighborhood U of t and $f^{\Delta}(t) \in \mathbf{R}^n$ such that for any $s \in U$ we have*

$$\|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\| \le \varepsilon(\sigma(t) - s),$$

and $f^{\Delta}(t)$ is called the derivative of f at t.

If f and g are differentiable at t, then the following equalities hold:

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t),$$
$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t).$$

The integral of a mapping f is always understood in Lebesgue's sense and written as $\int_s^t f(\tau)\Delta\tau$, for $s, t \in \mathbf{T}$. If $f \in C_{rd}(\mathbf{T}, \mathbf{R}^n)$ and $t \in \mathbf{T}$, then

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t). \tag{2.1}$$