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On Homomorphism of Valuation Algebras*

Guan Xue-chong[1,2] and Li Yong-ming[3]

(*1. College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710062*)

(*2. School of Mathematical Science, Xuzhou Normal University, Xuzhou, Jiangsu, 221116*)

(*3. College of Computer Science, Shaanxi Normal University, Xi'an, 710062*)

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Abstract: In this paper, firstly, a necessary condition and a sufficient condition for an isomorphism between two semiring-induced valuation algebras to exist are presented respectively. Then a general valuation homomorphism based on different domains is defined, and the corresponding homomorphism theorem of valuation algebra is proved.

Key words: homomorphism, valuation algebra, semiring

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1 Introduction

Valuation algebra is an abstract formalization from artificial intelligence including constraint systems (see [1]), Dempster-Shafer belief functions (see [2]), database theory, logic, and etc. There are three operations including labeling, combination and marginalization in a valuation algebra. With these operations on valuations, the system could combine information, and get information on a designated set of variables by marginalization. With further research and theoretical development, a new type of valuation algebra named domain-free valuation algebra is also put forward.

As valuation algebra is an algebraic structure, the concept of homomorphism between valuation algebras, which is derived from the classic study of universal algebra, has been defined naturally. Meanwhile, recent studies in [3], [4] have showed that valuation algebras induced by semirings play a very important role in applications. Based on the above fundamental factors, in this paper, we study the relationship between homomorphism of semiring-induced valuation algebras and that of semirings. Furthermore, in view of the lim-

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itation in the notion of valuation homomorphism, we give a general definition of valuation homomorphism which are based on different domain sets, and study its properties.

2 Notations and Preliminaries

The fundamental elements of a valuation algebra are valuations. In general, a valuation is a function that provides possible elements of a field for variables. Here a valuation represents some knowledge and information which may be a function, tuple or symbol.

In this study, variables will be designated by capital like X, Y, $\cdots$. The symbol Ω_X which contains at least two elements is the finite set of possible values of X, and it is called the frame of X. Lower-case letters such as x, y, $\cdots$, denote sets of variables. For a nonempty set s of variables, let Ω_s denote the Cartesian product of the frames Ω_X of the variables $X \in s$, i.e., $\Omega_s = \prod\limits_{X\in s} \Omega_X$, and Ω_s is called the frame of the set of variables s. If s is empty, for convenience, we denote $\Omega_\emptyset = \{\diamond\}$. We use lower-case, bold-faced letters such as $\mathbf{x}$, $\mathbf{y}$, $\cdots$ to designate the elements of Ω_s. If $\mathbf{x} \in \Omega_s$ and $t \subseteq s$, then $\mathbf{x}^{\downarrow t}$ denotes the projection of $\mathbf{x}$ to the subdomain t. In particular, we have $\mathbf{x}^{\downarrow\emptyset} = \diamond$.

Definition 2.1[1,3] *A semiring is a tuple* $\mathcal{A} = \langle A, +, \times, 0, 1\rangle$ *such that*

1. $+$, $\times$ *are commutative and associative*;
2. $\times$ *distributes over* $+$*;*
3. 0 *and* 1 *are the unit element of* $+$ *and* $\times$ *respectively. In addition,* 0 *is an absorbing element of* $\times$*, i.e.,*

$$0 \times a = a \times 0 = 0, \qquad a \in A.$$

For convenience, a semiring $\mathcal{A}$ will be denoted as A. If a semiring A satisfies

$$a + 1 = 1, \qquad a \in A,$$

then we call A a c-semiring. In a c-semiring A, we have

$$a + b = \sup\{a, b\}, \quad a \times b \leq a, \qquad a, b \in A.$$

Now we consider a non-empty finite set r of variables with finite frames and a semiring A. A semiring valuation ϕ with domain $s \subseteq r$ is defined to be a function that associates a value from A with $\mathbf{x} \in \Omega_s$, i.e., $\phi : \Omega_s \to A$. The symbol $d(\phi)$ denotes the domain of the valuation ϕ, i.e., $d(\phi) = s$. Φ_s denotes the set of all valuations with domain s, and $\Phi = \bigcup\limits_{s\subseteq r} \Phi_s$. Let $D = \mathcal{P}(r)$ denote the lattice of subset of r. We define the operations in the pair (Φ, D) by using the operations $+$ and $\times$ in A:

(1) Combination: $\otimes : \Phi \times \Phi \to \Phi$: for $\phi, \psi \in \Phi$ and $\mathbf{x} \in \Omega_{d(\phi)\cup d(\psi)}$ we define

$$\phi \otimes \psi(\mathbf{x}) = \phi(\mathbf{x}^{\downarrow d(\phi)}) \times \psi(\mathbf{x}^{\downarrow d(\psi)});$$

(2) Marginalization: $\downarrow$: $\Phi \times D \to \Phi$ is defined by

$$\phi^{\downarrow t}(\mathbf{x}) = \sum_{\mathbf{z}\in\Omega_{d(\phi)}:\mathbf{z}^{\downarrow t}=\mathbf{x}} \phi(\mathbf{z}), \qquad \phi \in \Phi,\ t \subseteq d(\phi),\ \mathbf{x} \in \Omega_t.$$

In [3], it has shown that (Φ, D) satisfies the following axioms (1)–(6) in Definition 2.2, if A is a semiring. The system (Φ, D) is called a valuation algebra induced by the semiring A.