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A Lower Bound of the Genus of a Self-amalgamated 3-manifolds*

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Abstract: Let M be a compact connected oriented 3-manifold with boundary, $Q_1, Q_2 \subset \partial M$ be two disjoint homeomorphic subsurfaces of ∂M, and $h : Q_1 \to Q_2$ be an orientation-reversing homeomorphism. Denote by M_h or $M_{Q_1=Q_2}$ the 3-manifold obtained from M by gluing Q_1 and Q_2 together via h. M_h is called a self-amalgamation of M along Q_1 and Q_2. Suppose Q_1 and Q_2 lie on the same component F' of $\partial M'$, and $F' - Q_1 \cup Q_2$ is connected. We give a lower bound to the Heegaard genus of M when M' has a Heegaard splitting with sufficiently high distance.

Key words: self-amalgamation, distance, Heegaard genus

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1 Introduction

Let M be a compact connected oriented 3-manifold with boundary, $Q_1, Q_2 \subset \partial M$ be two disjoint homeomorphic subsurfaces of ∂M, and $h : Q_1 \to Q_2$ be an orientation-reversing homeomorphism. Denote by M_h or $M_{Q_1=Q_2}$ the 3-manifold obtained from M by gluing Q_1 and Q_2 together via h. M_h is called a self-amalgamation of M along Q_1 and Q_2. Usually, $Q = Q_1 = Q_2$ is a non-separating surface properly embedded in M_h, and M can be reobtained from M_h by cutting M_h open along Q.

An interesting problem is how the genus of M_h is related to that of M. Here are partial related results:

Theorem 1.1[1] *Let M be a compact orientable 3-manifold, and Q a non-separating incompressible closed surface in M. Let M' be the 3-manifold obtained by cutting M open along Q. Suppose M' admits a Heegaard splitting $V' \cup_{S'} W'$ with $d(S') \geq 2g(M')$. Then $g(M) \geq g(M') - g(F)$.*

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Theorem 1.2[2] *Let M be a closed orientable 3-manifold, and Q a non-separating incompressible closed surface in M. Let M' be the 3-manifold obtained by cutting M open along Q. Suppose M' admits a Heegaard splitting $V' \cup_{S'} W'$ relative to $\partial M'$ with $d(S') > 2(g(M', \partial M') + 2g(Q))$. Then M has a unique minimal Heegaard splitting, i.e., the self-amalgamation of $V' \cup_{S'} W'$.*

Both Theorems 1.1 and 1.2 deal with the case in which the non-separating surface is closed. In the present paper, we consider the situation in which the non-separating surface is with boundary. We obtain a lower bound of the genus of the self-amalgamated 3-manifold under some condition in terms of distances of the previous Heegaard splittings.

The paper is organized as follows. In Section 2, we review some preliminaries which is used later. The statement of the main result and its proof are included in Section 3. All 3-manifolds in this paper are assumed to be compact and orientable.

2 Preliminaries

In this section, we review some fundamental facts on surfaces in 3-manifolds. Definitions and terms which have not been defined are all standard, and the reader is referred to, for example, [3].

A Heegaard splitting of a 3-manifold M is a decomposition

$$M = V \cup_S W$$

in which V and W are compression bodies such that

$$V \cap W = \partial_+ V = \partial_+ W = S$$

and

$$M = V \cup W.$$

S is called a Heegaard surface of M. The genus $g(S)$ of S is called the genus of the splitting $V \cup_S W$. We use $g(M)$ to denote the Heegaard genus of M, which is equal to the minimal genus of all Heegaard splittings of M. A Heegaard splitting $V \cup_S W$ for M is minimal if $g(S) = g(M)$. $V \cup_S W$ is said to be weakly reducible (see [4]) if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $V \cup_S W$ is strongly irreducible.

Let

$$M = V \cup_S W$$

be a Heegaard splitting, α and β be two essential simple closed curves in S. The distance $d(\alpha, \beta)$ of α and β is the smallest integer $n \geq 0$ such that there is a sequence of essential simple closed curves

$$\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$$

in S with $\alpha_{i-1} \cap \alpha_i = \emptyset$, for $1 \leq i \leq n$. The distance of the Heegaard splitting $V \cup_S W$ is defined to be

$$d(S) = \min \{d(\alpha, \beta)\},$$