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Hájek-Rényi-type Inequality for a Class of Random Variable Sequences and Its Applications*

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Abstract: In this paper, we obtain the Hájek-Rényi-type inequality for a class of random variable sequences and give some applications for associated random variable sequences, strongly positive dependent stochastic sequences and martingale difference sequences which generalize and improve the results of Prakasa Rao and Soo published in Statist. Probab. Lett., 57(2002) and 78(2008). Using this result, we get the integrability of supremum and the strong law of large numbers for a class of random variable sequences.

Key words: Hájek-Rényi-type inequality, associated random variable sequence, strongly positive dependent stochastic sequence, martingale difference sequence

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1 Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega,\ \mathcal{F},\ P)$,

$$S_n = \sum_{i=1}^{n}(X_i - EX_i), \quad n \geq 1, \qquad S_0 = 0,$$

and $\{b_n,\ n \geq 1\}$ be a nondecreasing sequence of positive numbers. Hájek and Rényi[1] proved that: If $\{X_n,\ n \geq 1\}$ is a sequence of independent random variables with finite

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second moment, then for any $\varepsilon > 0$ and $m < n$,

$$P\left\{\max_{m\le j\le n}\left|\frac{1}{b_j}\sum_{i=1}^{j}(X_i - EX_i)\right| \ge \varepsilon\right\} \le \frac{1}{\varepsilon^2}\left\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2} + \sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}\right\}. \tag{1.1}$$

Hájek-Rényi-type inequality has been studied by many authors; one can refer to [2]–[9]. In this paper, we study the Hájek-Rényi-type inequality under the general condition A1 below. In addition, we give some applications of Hájek-Rényi-type inequality which generalize and improve the results of Prakasa Rao[6] and Soo[9]. Let n and m be integers and C be a positive constant not depending on n and m in what follows.

A1 For any positive integers $m \le n$,

$$E\left\{\max_{m\le i\le n}\left|\sum_{j=m}^{i}(X_j - EX_j)\right|^2\right\} \le C\cdot E\left\{\sum_{j=m}^{n}(X_j - EX_j)\right\}^2, \tag{1.2}$$

$$Cov(X_i,\ X_j) \ge 0, \qquad i, j = 1, 2, \cdots \tag{1.3}$$

Lemma 1.1([5], Theorem 1.1) *Let $\beta_1, \beta_2, \cdots, \beta_n$ be a nondecreasing sequence of positive numbers, and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be nonnegative numbers. Let r be a fixed positive number. Assume that for each m with $1 \le m \le n$,*

$$E\left(\max_{1\le l\le m}\left|\sum_{j=1}^{l}X_j\right|\right)^r \le \sum_{l=1}^{m}\alpha_l. \tag{1.4}$$

Then

$$E\left(\max_{1\le l\le n}\left|\frac{\sum_{j=1}^{l}X_j}{\beta_l}\right|\right)^r \le 4\sum_{l=1}^{n}\frac{\alpha_l}{\beta_l^r}. \tag{1.5}$$

2 Hájek-Rényi-type Inequality

Theorem 2.1 *Let $\{X_n,\ n \ge 1\}$ be a sequence of random variables satisfying A1 and $\{b_n,\ n \ge 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and $n \ge 1$,*

$$P\left\{\max_{1\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \ge \varepsilon\right\} \le \frac{4C}{\varepsilon^2}\left\{\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2} + 2\sum_{1\le k<j\le n}\frac{Cov(X_k, X_j)}{b_j^2}\right\}, \tag{2.1}$$

where C is defined in (1.2).

Proof. Without loss of generality, we assume that $b_n \ge 1$. Let $\alpha = \sqrt{2}$. For $i \ge 0$, define

$$A_i = \{1 \le k \le n :\ \alpha^i \le b_k < \alpha^{i+1}\}.$$

For $A_i \ne \emptyset$, let

$$v(i) = \max\{k :\ k \in A_i\},$$

and t_n be the index of the last nonempty set A_i. Obviously,

$$A_iA_j = \emptyset, \qquad i \ne j$$

and

$$\sum_{i=0}^{t_n} A_i = \{1,\ 2,\ \cdots,\ n\}.$$