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On Weakly Semicommutative Rings*

Chen Wei-xing and Cui Shu-ying
(*School of Mathematics and Information Science, Shandong Institute of Business and Technology, Yantai, Shandong, 264005*)

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Abstract: A ring R is said to be weakly semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb \subseteq \mathrm{Nil}(R)$, where $\mathrm{Nil}(R)$ is the set of all nilpotent elements in R. In this note, we clarify the relationship between weakly semicommutative rings and NI-rings by proving that the notion of a weakly semicommutative ring is a proper generalization of NI-rings. We say that a ring R is weakly 2-primal if the set of nilpotent elements in R coincides with its Levitzki radical, and prove that if R is a weakly 2-primal ring which satisfies α-condition for an endomorphism α of R (that is, $ab = 0 \Leftrightarrow a\alpha(b) = 0$ where $a, b \in R$) then the skew polynomial ring $R[x; \alpha]$ is a weakly 2-primal ring, and that if R is a ring and I is an ideal of R such that I and R/I are both weakly semicommutative then R is weakly semicommutative. Those extend the main results of Liang *et al.* 2007 (Taiwanese J. Math., 11(5)(2007), 1359–1368) considerably. Moreover, several new results about weakly semicommutative rings and NI-rings are included.

Key words: weakly semicommutative ring, weakly 2-primal ring, NI-ring, Armendariz ring

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1 Introduction

Throughout this note all rings are associative with identity unless otherwise stated, and homomorphisms of rings preserve the identity. Given a ring R, we use the symbol $\mathrm{Nil}(R)$ to denote the set of all nilpotent elements in R, and $T_n(R)$ the ring of upper triangular matrices over R. The symbol $\mathrm{Nil}_*(R)$ denotes the prime radical of a ring R, $\mathrm{Nil}^*(R)$ its upper nil-radical, and L-rad(R) its Levitzki radical, respectively. For a nonempty subset S of a ring R, the symbol $\langle S\rangle$ stands for the subring (may not with 1) generated by S.

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Recall that a ring R is reduced if it has no nonzero nilpotent elements. A ring R is 2-primal if Nil(R) =Nil$_*$(R), a ring R is locally 2-primal if every finitely generated subring of R is 2-primal, and a ring R is an NI-ring if Nil(R) =Nil*(R) (see [1]). A ring R is called Armendariz if for any $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x]$ satistying $f(x)g(x) = 0$ it halds that $a_i b_j = 0$ for all i and j, a ring R is nil-Armendariz if $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) \in \text{Nil}(R)[x]$, then all $a_i b_j \in \text{Nil}(R)$ (see [2]), and a ring R is power-serieswise Armendariz if for $f(x) = \sum\limits_{i=0}^{\infty} a_i x^i$, $g(x) = \sum\limits_{j=0}^{\infty} b_j x^j \in R[[x]]$ satisfying $f(x)g(x) = 0$, it holds that $a_i b_j = 0$ for all i and j (see [3]).

A ring R is said to be semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. It is known that reduced $\Rightarrow$ semicommutative $\Rightarrow$ 2-primal $\Rightarrow$ locally 2-primal $\Rightarrow$ weakly 2-primal $\Rightarrow$ NI, and no reversal holds (see [1], [4] and Example 3.1 below). Historically, some of the earliest results known to us about semicommutative rings is due to Shin[5]. There are many papers to investigate semicommutative rings and their generalizations (see [6]–[11]). Liang *et al.*[11] call a ring R to be weakly semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb \subseteq \text{Nil}(R)$. The notion is a proper generalization of semicommutative rings by Example 2.2 of [11]. It is proved there that if R is a ring and $I \subseteq \text{Nil}(R)$ an ideal of R such that R/I is weakly semicommutative, then R is weakly semicommutative (see [11], Proposition 3.2). This implies that NI-rings are weakly semicommutative. It is natural to ask whether there is a weakly semicommutative ring which is not an NI-ring. We give a positive answer to this question later. The main results in [11] are as follows: (1) Let R be a semicommutative ring with an endomorphism α. If R satisfies α-condition, that is, $ab = 0 \Leftrightarrow a\alpha(b) = 0$ for $a, b \in R$, then the skew polynomial ring $R[x; \alpha]$ is weakly semicommutative; (2) If R is a ring and I an ideal of R such that R/I is weakly semicommutative and I is semicommutative, then R is weakly semicommutative. The main objective of this note is to extend the above results to more general cases. We call a ring R to be weakly 2-primal if Nil(R) =L-rad(R). It is proved that if R is a weakly 2-primal ring and satisfies α-condition for an endomorphism α of R, then the skew polynomial ring $R[x; \alpha]$ is weakly 2-primal and hence weakly semicommutative, and that if R is a ring and I an ideal of R such that I and R/I are weakly semicommutative, then R is weakly semicommutative. Moreover, several new results about weakly semicommutative rings and NI-rings are obtained.

2 Weakly Semicommutative Rings and NI-rings

In this section we clarify the relationship between weakly semicommutative rings and NI-rings, and study several further properties of weakly semicommutative rings and NI-rings. There are many characterizations of NI-rings in [10]. To distinguish weakly semicommutative rings from NI-rings clearly, we start by giving some new characterizations of NI-rings.