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On the Gracefulness of Graph $(jC_{4n}) \cup P_m$*

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Abstract: The present paper deals with the gracefulness of unconnected graph $(jC_{4n}) \cup P_m$, and proves the following result: for positive integers n, j and m with $n \geq 1$, $j \geq 2$, the unconnected graph $(jC_{4n}) \cup P_m$ is a graceful graph for $m = j - 1$ or $m \geq n + j$, where C_{4n} is a cycle with $4n$ vertexes, P_m is a path with $m + 1$ vertexes, and $(jC_{4n}) \cup P_m$ denotes the disjoint union of $j - C_{4n}$ and P_m.

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1 Introduction

Most graceful graphs and graceful labeling methods trace their origin to a work by Rosa[1] in 1967. However, the term "graceful labeling" were not used until Golomb gave such labelings several years later (see [2]). The graceful labelings were introduced by Gallian[3]. The gracefulness of graphs is connected with the graceful labeling. Dong[4] proved that the graph $C_{4n} \cup C_{4n} \cup P_m$ is graceful for $4n \leq m \leq 4n + 2$. On the basis of these theories, we prove that the graph $(jC_{4n}) \cup P_m$, which denotes the disjoint union of $j - C_{4n}$ and P_m, is graceful for all n and $m = j - 1$ or $m \geq j + n$. When studying graceful labelings, we only consider simple graphs, i.e., graphs without loops or parallel edges.

2 Preliminaries

Definition 2.1 [2] *A graph $G(V, E)$ is called graceful if for any $v \in V$ there exists a nonnegative integer $f(v)$ satisfying the following conditions*:

(1) $\max\{f(v) | v \in V\} = |E(G)|$;

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(2) *for any* $u, v \in V$ *with* $u \neq v$, $f(u) \neq f(v)$;

(3) *for any* $e_1, e_2 \in E(G)$ *with* $e_1 \neq e_2$, $f'(e_1) \neq f'(e_2)$,

where $f(v)$ *is called to the labeling of the vertex* v, f *is a graceful value (or called graceful labeling),* $f'(e) = |f(u) - f(v)|$, *and* $uv = e$ *are called edge labelings.*

Definition 2.2 [5] *Let* f *be a graceful labeling of the graceful bipartite graph* $G = (X, Y, E)$. *If* $\max\limits_{u \in X}\{f(u)\} < \min\limits_{v \in Y}\{f(v)\}$, *then* G *is an alternating graph.*

Lemma 2.1 [5] *The cycle* C_{4n} *is a graceful graph and the graceful labeling* θ *is as follows*:

$$\begin{cases} \theta(x_{2k}) = 4n - k + 1, & k = 1, 2, \cdots, 2n; \\ \theta(x_{2k-1}) = k - 1, & k = 1, 2, \cdots, n; \\ \theta(x_{2k-1}) = k, & k = n+1, n+2, \cdots, 2n. \end{cases} \tag{2.1}$$

Lemma 2.2 [6] *Let* $P_m = v_0 v_1 \cdots v_m$ *be a path of length* m. *If* m *is an odd number, then there exists an alternating labeling* f *of* P_m *such that* $f(v_0) = k$ *for any* $k \in \{0, 1, \cdots, m\}$. *If* m *is an even number, then there exists an alternating labeling* f *of* P_m *such that* $f(v_0) = k$ *for any* $k \in \{0, 1, \cdots, m\}$ *and* $k \neq \dfrac{m}{4}, \ \dfrac{3m}{4}$.

Lemma 2.3 *Let* $P_{4k} = v_0 v_1 \cdots v_{4k}$ *be a path. If* $m = 4k$, *then there exists a graceful labeling* f *of* P_{4k} *such that* $f(v_0) = k = \dfrac{m}{4}$, *and there exists a graceful labeling* g *of* P_{4k} *such that* $g(v_0) = 3k$.

Proof. Put a vertex labeling f of the path $P_{4k} = v_0 v_1 \cdots v_{4k}$ as follows:

$$\begin{cases} f(v_{2n}) = k - n, & n = 0, 1, \cdots, k; \\ f(v_{2n}) = k + n, & n = k+1, k+2, \cdots, 2k; \\ f(v_{2n-1}) = 3k + n, & n = 1, 2, \cdots, k; \\ f(v_{2n-1}) = 3k - n + 1, & n = k+1, k+2, \cdots, 2k. \end{cases} \tag{2.2}$$

Clearly, f is a graceful labeling of the path $P_{4k} = v_0 v_1 \cdots v_{4k}$ such that $f(v_0) = k$ holds. Let

$$g(u) = |E(P_{4k})| - f(u) = 4k - f(u), \qquad u \in V(P_{4k}).$$

Then g is another graceful labeling such that

$$g(v_0) = 4k - f(v_0) = 3k.$$

From Lemmas 2.2 and 2.3 we have

Lemma 2.4 *Let* $P_m = v_0 v_1 \cdots v_m$ *be a path. Then there exists a graceful labeling* f *such that* $f(v_0) = n$ *for any* $n \in \{0, 1, \cdots, m\}$.

3 The Gracefulness of Graph $(jC_{4n}) \cup P_m$

Theorem 3.1 *Let* $n \geq 1$, $j \geq 2$. *Then the graph* $(jC_{4n}) \cup P_m$ *is graceful for* $m = j - 1$ *or* $m \geq n + j$.