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Rational Invariants of the Generalized Classical Groups*

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Abstract: In this paper, we give transcendence bases of the rational invariants fields of the generalized classical groups and their subgroups B, N and T, and we also compute the orders of them. Furthermore, we give explicit generators for the rational invariants fields of the Borel subgroup and the Neron-Severi subgroup of the general linear group.

Key words: generalized classical group, BN-pair, invariant, finite field

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1 Introduction

Invariants are the important objects, used to study the geometric properties of the groups, and they have been extensively studied (see, for example, [1]–[5]). An important Noether's problem asks whether the invariant field $K(X_1, \cdots, X_n)^G$ is purely transcendental over K. When K is a finite field, Dickson[1] answered the question for the general linear group by giving explicit generators. Then Carlisle and Kropholler[2] found explicit generators for $F_{q^2}(X_1, \cdots, X_n)^{U_n(F_{q^2})}$ and $F_q(X_1, \cdots, X_n)^{O_n(F_q,Q)}$.

Definition 1.1[6] (1) *Let F_q be a finite field and H be an invertible skew-symmetric matrix. The generalized symplectic group is defined to be*

$$GSp_{2\nu}(F_q, H) = \{P \in GL_n(F_q) \mid PHP^t = \lambda H \textit{ for some } \lambda \in F_q^*\}.$$

Without being mentioned explicitly, we often omit to mention the form of the invertible skew-symmetric matrix H and $GSp_{2\nu}(F_q, H)$ is just simply denoted by $GSp_{2\nu}(F_q)$.

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(2) *Let* F_q *be a finite field with* $\mathrm{char}F_q \neq 2$ *and* H *be an invertible symmetric matrix. The generalized orthogonal group is defined to be*

$$GO_n(F_q, H) = \{P \in GL_n(F_q, H) \mid PHP^t = \lambda H \ \textit{for some}\ \lambda \in F_q^*\},$$

and it is simply denoted by $GO_n(F_q)$.

(3) *Let* F_{q^2} *be a finite field with* $\mathrm{char}F_{q^2} \neq 2$. F_{q^2} *has an involutive automorphism, i.e., an automorphism of order* 2, *which is given by*

$$a \longmapsto \bar{a} = a^q,$$

and the fixed field of this automorphism is F_q. *Let* H *be an invertible matrix such that* $H^t = \bar{H}$. *The generalized unitary group is defined to be*

$$GU_n(F_{q^2}, H) = \{P \in GL_n(F_q) \mid \bar{P}HP^t = \lambda H \ \textit{for some}\ \lambda \in F_q^*\},$$

which is simply denoted by $GU_n(F_{q^2})$.

Next, we discribe the actions of matrix on the rational functions.

Let F_q be a finite field with $\mathrm{char}F_q = p$ and $GL_n(F_q)$ be the general linear group. For any $T \in GL_n(F_q)$, T induces an F_q-linear action σ_T on the rational function field which is defined by

$$\sigma_T(f(X_1, \cdots, X_n)) = f(\sigma_T(X_1), \cdots, \sigma_T(X_n)), \qquad f(X_1, \cdots, X_n) \in F_q(X_1, \cdots, X_n),$$

where

$$\sigma_T(X_i) = t_{i1}X_1 + t_{i2}X_2 + \cdots + t_{in}X_n, \qquad T = (t_{ij}),\ i, j = 1, 2, \cdots, n.$$

We have known that (see [7])

$$F_q(X_1, \cdots, X_n)^{Sp_n(F_q, H)} = F_q(P_{n1}, \cdots, P_{nn}),$$

where

$$P_{nk} = \sum_{1 \leq i,j \leq n} a_{ij} X_i X_j^{q^k},$$

$$H = (a_{ij}), \quad H^t = -H, \qquad 1 \leq i, j \leq n,\ k = 1, 2, \cdots, n;$$

$$F_q(X_1, \cdots, X_n)^{O_n(F_q, H)} = F_q(Q_{n0}, \cdots, Q_{n,n-1}),$$

where

$$Q_{nk} = \sum_{1 \leq i,j \leq n} c_{ij} X_i X_j^{q^k},$$

$$H = (c_{ij}), \quad H^t = H, \qquad 1 \leq i, j \leq n,\ k = 0, 1, \cdots, n-1,$$

and

$$F_{q^2}(X_1, \cdots, X_n)^{U_n(F_{q^2}, H)} = F_q(R_{n0}, \cdots, R_{n,n-1}),$$

where

$$R_{nk} = \sum_{1 \leq i,j \leq n} d_{ij} X_i^{p^l} X_j^{q^k},$$

$$H = (d_{ij}), \quad H^t = \bar{H}, \qquad 1 \leq i, j \leq n,\ k = 0, 1, \cdots, n-1.$$

We consider the rational invariants of the generalized classical groups and their subgroups B, N and T.