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Projections, Birkhoff Orthogonality and Angles in Normed Spaces*

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Abstract: Let X be a Minkowski plane, i.e., a real two dimensional normed linear space. We use projections to give a definition of the angle $A_q(\boldsymbol{x},\ \boldsymbol{y})$ between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in X, such that $\boldsymbol{x}$ is Birkhoff orthogonal to $\boldsymbol{y}$ if and only if $A_q(\boldsymbol{x},\ \boldsymbol{y}) = \dfrac{\pi}{2}$. Some other properties of this angle are also discussed.

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1 Introduction and Preliminaries

In the research of geometry in inner spaces, orthogonality plays a very important role. In general normed spaces, many new orthogonalities have been introduced, such as Birkhoff orthogonality in [1], isosceles orthogonality in [2] and so on. These definitions of orthogonalities are different, and their relations were discussed in [3]. In 1993, Milicic[4] introduced g-orthogonality in normed spaces via Gateaux derivatives. In [5], it is shown that the angle $A(\boldsymbol{x},\ \boldsymbol{y})$ in X satisfies the basic properties. In [6] and [7], there are more about the geometry of Minkowski plane. James[8] gave a result of Birkhoff orthogonality in l_2^∞.

Birkhoff orthogonality was introduced by Birkhoff[1] in 1935, which is the first notion of orthogonality in normed spaces.

Definition 1.1 *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two vectors in a normed space X. $\boldsymbol{x}$ is said to be Birkhoff orthogonal to $\boldsymbol{y}$, denoted by $\boldsymbol{x}\perp_B\boldsymbol{y}$, if for any $t\in\mathbf{R}$*

$$\|\boldsymbol{x}+t\boldsymbol{y}\| \geq \|\boldsymbol{x}\|.$$

In 1993, Milicic[4] introduced g-orthogonality in normed spaces via Gateaux derivatives. In fact, one has the notion of g-angle related to g-orthogonality.

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Definition 1.2 *The functional $g : X^2 \to \mathbf{R}$ is defined by*

$$g(\boldsymbol{x},\ \boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{x}\|(\tau_+(\boldsymbol{x},\ \boldsymbol{y}) + \tau_-(\boldsymbol{x},\ \boldsymbol{y})),$$

where

$$\tau_\pm(\boldsymbol{x},\ \boldsymbol{y}) = \lim_{t\to\pm 0} \frac{\|\boldsymbol{x}+t\boldsymbol{y}\| - \|\boldsymbol{x}\|}{t}.$$

The g-angle between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, denoted by $A_g(\boldsymbol{x},\ \boldsymbol{y})$, is given by

$$A_g(\boldsymbol{x},\ \boldsymbol{y}) = \arccos\frac{g(\boldsymbol{x},\ \boldsymbol{y})}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|}.$$

Furthermore, $\boldsymbol{x}$ is said to be g-orthogonal to $\boldsymbol{y}$, denoted by $\boldsymbol{x}\perp_g\boldsymbol{y}$, if

$$g(\boldsymbol{x},\ \boldsymbol{y}) = 0,$$

i.e.,

$$A_g(\boldsymbol{x},\ \boldsymbol{y}) = \frac{\pi}{2}.$$

In an inner product space $(X,\ \langle\ \cdot\ ,\ \cdot\ \rangle)$, the angle $A(\boldsymbol{x},\ \boldsymbol{y})$ between two nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in X is usually given by

$$A(\boldsymbol{x},\ \boldsymbol{y}) = \arccos\frac{\langle\boldsymbol{x},\ \boldsymbol{y}\rangle}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|},$$

where $\|\boldsymbol{x}\| = \langle\boldsymbol{x},\ \boldsymbol{y}\rangle^{1/2}$ denotes the induced norm in X. One may observe that the angle $A(\boldsymbol{x},\ \boldsymbol{y})$ in X satisfies the following basic properties (see [5]):

⋄ Parallelism: $A(\boldsymbol{x},\ \boldsymbol{y}) = 0$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are of the same direction; $A(\boldsymbol{x},\ \boldsymbol{y}) = \pi$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are of opposite direction.

⋄ Symmetry: $A(\boldsymbol{x},\ \boldsymbol{y}) = A(\boldsymbol{y},\ \boldsymbol{x})$ for every $\boldsymbol{x}, \boldsymbol{y} \in X$.

⋄ Homogeneity:

$$A(a\boldsymbol{x},\ b\boldsymbol{y}) = \begin{cases} A(\boldsymbol{x},\ \boldsymbol{y}), & ab > 0; \\ \pi - A(\boldsymbol{x},\ \boldsymbol{y}), & ab < 0. \end{cases}$$

⋄ Continuity: If $\boldsymbol{x}_n \to \boldsymbol{x}$ and $\boldsymbol{y}_n \to \boldsymbol{y}$ (in norm), then $A(\boldsymbol{x}_n,\ \boldsymbol{y}_n) \to A(\boldsymbol{x},\ \boldsymbol{y})$.

The g-angle is identical with the usual angle in an inner space and has the following properties:

(I) Part of parallelism property: If $\boldsymbol{x}$ and $\boldsymbol{y}$ are of the same direction, then

$$A_g(\boldsymbol{x},\ \boldsymbol{y}) = 0;$$

if $\boldsymbol{x}$ and $\boldsymbol{y}$ are of opposite direction, then

$$A_g(\boldsymbol{x},\ \boldsymbol{y}) = \pi.$$

(II) Part of homogeneity property:

$$A_g(a\boldsymbol{x},\ b\boldsymbol{y}) = A_g(\boldsymbol{x},\ \boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in X,\ a, b \in \mathbf{R};$$

(III) Homogeneity property:

$$A_g(a\boldsymbol{x},\ b\boldsymbol{y}) = \begin{cases} A_g(\boldsymbol{x},\ \boldsymbol{y}), & ab > 0; \\ \pi - A_g(\boldsymbol{x},\ \boldsymbol{y}), & ab < 0. \end{cases}$$

(IV) Part of continuity property: If $\boldsymbol{y}_n \to \boldsymbol{y}$ (in norm), then $A_g(\boldsymbol{x},\ \boldsymbol{y}_n) \to A_g(\boldsymbol{x},\ \boldsymbol{y})$.

However, g-orthogonality is not equivalent to Birkhoff orthogonality. In this article, we use projections to give a definition of the angle $A_q(\boldsymbol{x},\ \boldsymbol{y})$ between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x}$ is Birkhoff orthogonal to $\boldsymbol{y}$ if and only if

$$A_q(\boldsymbol{x},\ \boldsymbol{y}) = \frac{\pi}{2}.$$