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Bombieri's Theorem in Short Intervals*

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Abstract: Under the assumption of sixth power large sieve mean-value of Dirichlet L-function, we improve Bombieri's theorem in short intervals by virtue of the large sieve method and Heath-Brown's identity.

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1 Introduction

Let $\Lambda(n)$ be the von Mangoldt function and

$$\psi(x, q; a) = \sum_{\substack{n \le x \\ n \equiv a (\bmod\ q)}} \Lambda(n).$$

The well-known theorem of Bombieri-Vinogradov theorem (see [1]) states that

$$\sum_{q \le Q} \max_{(a,q)=1} \max_{z \le x} \left| \psi(z, q; a) - \frac{z}{\varphi(q)} \right| \ll \frac{x}{(\ln x)^A}, \tag{1.1}$$

where A is an arbitrary positive constant, and

$$Q = \frac{x^{\frac{1}{2}}}{(\ln x)^B}$$

with

$$B = B(A) > 0.$$

The problem of finding a result analogous to (1.1) for short intervals was first investigated by Jutila[2]. By using the zero-density method he established

$$\sum_{q \le Q} \max_{(a,q)=1} \max_{h \le y} \max_{\frac{1}{2}x < z \le x} \left| \psi(z + h, q; a) - \psi(z, q; a) - \frac{h}{\varphi(q)} \right| \ll \frac{y}{(\ln x)^A}, \tag{1.2}$$

where

$$y = x^\theta$$

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and

$$Q = \frac{x^{\psi}}{(\ln x)^{B(A)}}$$

with

$$\psi < \frac{4c\theta + 2\theta - 1 - 4c}{6 + 4c},$$

$$c = \inf\left\{\xi : \zeta\left(\frac{1}{2} + \mathrm{i}t\right) \ll t^{\xi}\right\}.$$

Subsequently, a number of authors improved Jutila's result, showing that (1.2) holds for smaller y and/or larger values of Q (see, e.g., [3–8]). The best known results hitherto are

$$\psi \le \theta - \frac{1}{2}, \qquad \frac{3}{5} < \theta \le 1 \tag{1.3}$$

or

$$\psi \le \theta - \frac{11}{20}, \qquad \frac{7}{12} < \theta \le 1. \tag{1.4}$$

The former result is established independently by Perelli *et al.*[5] and Timofeev[8], while the latter result is due to Timofeev[8].

The Zero-Density Hypothesis would imply that (1.2) is true if

$$\psi \le \theta - \frac{1}{2}, \qquad \frac{1}{2} < \theta \le 1.$$

In this paper, under the assumption of sixth power large sieve mean-value of Dirichlet L-function, we are able to prove the following result.

Theorem 1.1 *The estimate* (1.2) *is true if*

$$\psi \le \theta - \frac{1}{2}, \qquad \frac{7}{12} < \theta \le 1.$$

We introduce some notations in the following.

As usual, $\varphi(n)$ stands for the function of Euler. $\chi \bmod q$ is a Dirichlet character modulo q, and $L(s,\chi)$ is the Dirichlet L-function attached to χ. The summation $\sum\limits_{\chi \bmod q}{}^{*}$ means that the summation is over primitive characters modulo q. $L = \ln x$ and $q \sim Q$ means that $\frac{Q}{2} < q < Q$.

$$d_k(m) = \sum_{m = m_1 m_2 \cdots m_k} 1, \qquad k = 1, 2, \cdots$$

2 Proof of Theorem 1.1

Let

$$X^{\frac{3}{7}} < Y \le X$$

and M_1, M_2, $\cdots$, M_{14} be positive real numbers such that

$$Y \le M_1 \cdots M_{14} < X \qquad \text{and} \qquad 2M_8, \cdots, 2M_{14} \le X^{1/7}. \tag{2.1}$$