Communications in Mathematical Research
28(3)(2012), 243–251

Blow-up of Solutions to Porous Medium Equations with a Nonlocal Boundary Condition and a Moving Localized Source*

Sun Peng, Gao Wen-jie and Han Yu-zhu
(*School of Mathematics, Jilin University, Changchun, 130012*)

Abstract: This paper is devoted to the blow-up properties of solutions to the porous medium equations with a nonlocal boundary condition and a moving localized source. Conditions for the existence of global or blow-up solutions are obtained. Moreover, we prove that the unique solution has global blow-up property whenever blow-up occurs.

Key words: blow-up, moving localized source, nonlocal boundary condition, global blow-up

2000 MR subject classification: 35K55, 35K57, 35K60, 35K65

Document code: A

Article ID: 1674-5647(2012)03-0243-09

1 Introduction

In this paper, we consider positive solutions to the following porous medium equations with a nonlocal boundary condition and a moving localized source:

$$\begin{cases} u_t = \Delta u^m + f(u(x_0(t), t)), & x \in \Omega, \ t > 0, \\ u(x, t) = \displaystyle\int_\Omega k(x, y)u(y, t)\mathrm{d}y, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $m > 1$ is a constant and Ω is a bounded domain in $\mathbf{R}^N$ $(N > 1)$ with smooth boundary $\partial\Omega$, $k(x, y) \not\equiv 0$ is a nonnegative continuous function on $x \in \partial\Omega$ and $y \in \bar{\Omega}$, $u_0(x)$ is a positive continuous function and satisfies the compatibility condition

$$u_0(x) = \int_\Omega k(x, y)u_0(y)\mathrm{d}y, \qquad x \in \partial\Omega,$$

$x_0(t)$ is a continuously differentiable function from $\mathbf{R}^+$ to a fixed compact subset of Ω denoted by K, and $f(s)$ satisfies the following assumptions:

(H1) $f(s) \in C[0, \infty) \cap C^1(0, \infty)$;

*Received date: June 9, 2010.
Foundation item: The NSF (10771085) of China, the Key Lab of Symbolic Computation and Knowledge Engineering of Ministry of Education and the 985 program of Jilin University.

(H2) $f(0) \geqslant 0$ and $f'(s) > 0$ on $(0, \infty)$.

There have been many articles dealing with the properties of solutions to the partial differential equations with local boundary conditions. However, there are some important phenomenas formulated into the parabolic equations that are coupled with nonlocal boundary conditions in mathematical modelling such as thermoelasticity theory (see [1–3]). In this case, the solution $u(x, t)$ describes entropy per volume material.

Our work is motivated by the following discussions on nonlocal problems. Consider the following parabolic problem with a nonlocal boundary condition:

$$\begin{cases} u_t = \Delta u + g(x, u), & x \in \Omega,\ t > 0, \\ u(x, t) = \displaystyle\int_\Omega k(x, y) u(y, t) \mathrm{d}y, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

Friedman[4] established the global existence of solutions and discussed the monotonic decay in the case of

$$g(x, u) = c(x)u$$

with

$$c(x) \leqslant 0, \quad \int_\Omega |k(x, y)| \mathrm{d}y < 1, \qquad x \in \partial\Omega.$$

Seo[5] gave a global upper bound of solutions to the problem (1.2) with

$$g(x, u) = cu, \qquad k(x, y) \geqslant 0,$$

where c is a constant without any restriction. And Seo[5] also discussed the decreasing property of boundary values in the case of

$$c \geqslant 0, \qquad \|k(x, \cdot)\|_1 > 1.$$

Deng[6] gave a comparison principle and local existence of a classical solution to the problem (1.2) with general $g(x, u)$. Furthermore, for

$$g(x, u) = c(x)u,$$

Deng[6] removed the restrictions

$$c(x) \leqslant 0, \qquad \int_\Omega |k(x, y)| \mathrm{d}y < 1$$

in [4] and showed that the solution exists globally and may increase at most exponentially with t. When $g(x, u)$ is superlinear, the solution to the problem (1.2) may blow up in finite time. In particular, Seo[7] investigated the problem (1.2) with

$$g(x, u) = g(u)$$

and gave a blow-up condition of the positive solutions by using the supersolution and subsolution methods. The blow-up rate estimates for

$$g(u) = u^p, \qquad g(u) = \mathrm{e}^u$$

were also derived. For more related works, the readers may refer to [8–10] and the references therein.

Recently, Wang *et al.*[11] studied the following porous medium problem with a nonlocal