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An Almost Sure Central Limit Theorem for Weighted Sums of Mixing Sequences*

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Abstract: In this paper, we prove an almost sure central limit theorem for weighted sums of mixing sequences of random variables without stationary assumptions. We no longer restrict to logarithmic averages, but allow rather arbitrary weight sequences. This extends the earlier work on mixing random variables.

Key words: almost sure central limit theorem, weighted sum, mixing sequence, logarithmic average

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1 Introduction and Main Results

The almost sure central limit theorem (ASCLT) was first introduced independently by Brosamler[1] and Schatte[2]. Since then many interesting results have been discovered in this field. Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d. random variables with

$$EX_1 = 0, \qquad EX_1^2 = 1, \qquad S_n = \sum_{i=1}^{n} X_i.$$

Then the ASCLT for $\{X_n,\ n \geq 1\}$ is described as for any $x \in \mathbf{R}$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\{S_k/\sqrt{k} \leq x\} = \Phi(x) \qquad \text{a.s.}$$

Here and in the sequel, $\log n = \ln(n \vee \mathrm{e})$, $I\{\ \cdot\ \}$ denotes indicator function and $\Phi(\ \cdot\ )$ is the distribution function of the standard normal random variable. The above ASCLT is said to be hold with logarithmic averages. Many interesting results have been obtained in this field (see [3–11]).

Many useful linear statistics based on a random sample are weighted sums of random variables. Examples include least-square estimators, nonparametric regression function es-

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timators and jackknife estimates, among others. Therefore, studies of the limit properties for these weighted sums have been demonstrated to be a significant process in probability theory with applications in mathematical statistics. Up to now, various limit properties such as central limit theorem, law of large number, complete convergence and so on, have been studied by many authors. Zhang *et al.*[8] obtained the ASCLT for weighted sums under association with logarithmic averages. But few results about ASCLT for weighted sums under mixing sequences have been studied. From Zhang *et al.*[9], Gonchigdanzan and Rempala[12], Li and Wang[13], we know that the proof of the ASCLT for weighted sums is an important step in the proof of the ASCLT for products of sums.

At first, we recall some measures of dependence between two algebras.

Definition 1.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras of events. We define*

$$\varphi(\mathcal{A},\mathcal{B}) = \sup_{A\in\mathcal{A}, B\in\mathcal{B}, P(A)\neq 0} |P(B|A) - P(B)|,$$
$$\rho(\mathcal{A},\mathcal{B}) = \sup_{f\in L_2(\mathcal{A}), g\in L_2(\mathcal{B})} |\mathrm{corr}(f,\ g)|$$

and

$$\alpha(\mathcal{A},\mathcal{B}) = \sup_{A\in\mathcal{A}, B\in\mathcal{B}} |P(AB) - P(A)P(B)|.$$

Definition 1.2 *Let $\{X_i\}$ be a stochastic sequence and $\mathcal{F}_n^m = \sigma(X_i,\ n \leq i \leq m)$.*

(a) *We call the sequence φ-mixing if $\varphi(n) \to 0$, where*
$$\varphi(n) = \sup_k \varphi(\mathcal{F}_1^k,\ \mathcal{F}_{k+n}^\infty).$$

(b) *We call the sequence ρ-mixing if $\rho(n) \to 0$, where*
$$\rho(n) = \sup_k \rho(\mathcal{F}_1^k,\ \mathcal{F}_{k+n}^\infty).$$

(c) *We call the sequence strongly mixing if $\alpha(n) \to 0$, where*
$$\alpha(n) = \sup_k \alpha(\mathcal{F}_1^k,\ \mathcal{F}_{k+n}^\infty).$$

It is well known that the φ-mixing condition is the most restrictive and the strong mixing is the weakly among all (see [14] for a survey). In this paper, we study the ASCLT of weighted sums for these three classes of dependent random variables. Our theorem extends the known results for strictly mixing sequence from equal weights to general weights, weakening at the same time the assumption of stationary. And also we extend the ASCLT with other sequence $\{D_n,\ n \geq 1\}$ tending faster to infinity than the logarithmic averages. The following theorem is our main result.

Theorem 1.1 *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables with $EX_n = 0$, and $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ be an array of real numbers satisfying the following conditions*:

(C1) $\sup\limits_n \sum\limits_{i=1}^n a_{ni}^2 < \infty$ *and* $|a_{ni}| \leq C\dfrac{1 \vee \log^\gamma(n/i)}{n^{1/2}}$, $1 \leq i \leq n$, $n \geq 1$ *for some* $\gamma > 0$;

(C2) $\mathrm{var}(S_n) \to 1$ *as* $n \to \infty$ *and* $\{X_n^2\}$ *is a uniformly integrable family, where* $S_n = \sum\limits_{i=1}^n a_{ni}X_i$.