COMMUNICATIONS IN MATHEMATICAL RESEARCH
29(1)(2013), 51–60

Dynamics and Long Time Convergence of the Extended Fisher-Kolmogorov Equation under Numerical Discretization*

WANG JUE
(*College of Science, Harbin Engineering University, Harbin, 150001*)

Communicated by Ma Fu-ming

Abstract: We present a numerical study of the long time behavior of approximation solution to the Extended Fisher–Kolmogorov equation with periodic boundary conditions. The unique solvability of numerical solution is shown. It is proved that there exists a global attractor of the discrete dynamical system. Furthermore, we obtain the long-time stability and convergence of the difference scheme and the upper semicontinuity $d(\mathcal{A}_{h,\tau}, \mathcal{A}) \to 0$. Our results show that the difference scheme can effectively simulate the infinite dimensional dynamical systems.

Key words: Extended Fisher–Kolmogorov equation, finite difference method, global attractor, long time stability and convergence

2000 MR subject classification: 65M06, 37L99, 35B40, 35Q55, 65M20

Document code: A

Article ID: 1674-5647(2013)01-0051-10

1 Introduction

The Extended Fisher-Kolmogorov (EFK) equation is given by

$$\frac{\partial u}{\partial t} + \beta\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + (u^3 - u) = 0, \qquad x \in \Omega,\ t \geq 0 \tag{1.1}$$

with the boundary condition

$$u(0,t) = u(x+L,t), \qquad x \in \Omega,\ t \geq 0 \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $\beta > 0$, $0 < L < +\infty$ and $\Omega = (0, L)$ is a bounded domain in $\mathbf{R}$ with boundary $\partial\Omega$, and u_0 is a given L-periodic function.

When $\beta = 0$ in (1.1), the standard Fisher-Kolmogorov equation was obtained (see [1–2]). Adding a stabilizing fourth order derivative term to the standard Fisher-Kolmogorov

*Received date: Aug. 24, 2010.
Foundation item: The NSF (10871055) of China and the Fundamental Research Funds (HEUCFL20111102) for the Central Universities.

equation, the equation (1.1) is proposed and called as Extended Fisher-Kolmogorov equation (see [3–6]).

The equation (1.1) occurs in a variety of applications such as pattern formation in bistable systems, propagation of domain walls in liquid crystals, travelling waves in reaction diffusion systems and mezoscopic model of a phase transition in a binary system near the Lipschitz point (see [4, 7–9]). In particular, in the phase transitions near critical points (Lipschitz points), the higher order gradient terms in the free energy functional can no longer be neglected and the fourth order derivative becomes important.

There have been a number of papers in the literature dealing with equations similar to the equation (1.1) (see [10–13]). In recent years, attention has been focused on the connection between finite-dimensional dynamical system theory and the long-time behavior of solutions of a priori infinite-dimensional dynamical systems described by partial differential equations. In particular, the techniques have been developed to establish this connection in a rigorous and quantitative way by showing how the dimension of the global attractor may be estimated for some dissipative partial differential equations (see [14–17]). The long-time behavior of the solutions to (1.1) is studied theoretically in [18].

For the long-time computation of partial differential equations, the error estimate are important in both space and time directions. Simo and Armero[19] pointed out that the first order scheme with long time stability and convergence is more effective than the second order scheme. Recently, some useful results about equivalence of equi-attraction and continuous convergence of attractors in different spaces have been given in [20]. To compute a trajectory numerically, long-time computation generally suffers from error accumulation at the unavoidable exponential rates. A numerical trajectory eventually leaves the exact trajectory and no longer shows any information about the original trajectory. On the other hand, for dissipative system such as the EFK equation, if the discretization schemes are appropriately selected, the numerical trajectory is expected to approach a discrete attractor and it eventually enters and stays in a small neighborhood of the attractor.

For this reason, we consider the error estimates for a global attractor. Existence of attractors for the dissipative systems is proved. The remainder of this paper is organized as follows. In Section 2, we describe a new finite difference scheme for the EFK equation and prove that the difference scheme is uniquely solvable. In Section 3, we derive the priori error estimates for numerical solution to obtain the existence of a global attractor. In Section 4, we discuss the long time stability and convergence of the difference scheme and the upper semicontinuity $d(\mathcal{A}_{h,\tau}, \mathcal{A}) \to 0$.

2 Finite Difference Scheme and Unique Solvability of Difference Approximation

Let $h = L/J$ be the uniform step size in the spatial direction for a positive integer J. Let τ denote the uniform step size in the temporal direction. Denote $V_i^n = V(x_i, t_n)$ for $t_n = n\tau$,