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Existence of Positive Solutions for Higher Order Boundary Value Problem on Time Scales*

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Abstract: In this paper, we investigate the existence of positive solutions of a class higher order boundary value problems on time scales. The class of boundary value problems educes a four-point (or three-point or two-point) boundary value problems, for which some similar results are established. Our approach relies on the Krasnosel'skii fixed point theorem. The result of this paper is new and extends previously known results.

Key words: higher order boundary value problem, positive solution, semipositone, on time scale, fixed point
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1 Introduction

In this paper, we study the existence of positive solutions of higher order boundary value problem (BVP) on time scales as follows:

$$x^{(n)}(t) + f(t, x(t)) = 0, \qquad a < t < b, \tag{1.1}$$

$$x(a) = h\Big(\int_a^b x(t)\mathrm{d}\varphi(t)\Big),\ x'(a) = 0,\ \cdots,\ x^{(n-2)}(a) = 0,\ x(b) = g\Big(\int_a^b x(t)\mathrm{d}\phi(\mathrm{t})\Big), \tag{1.2}$$

where $\int_a^b x(t)\mathrm{d}\varphi(\mathrm{t})$, $\int_a^b x(t)\mathrm{d}\phi(\mathrm{t})$ denote the Riemann-Stieltjes integrals.

We assume that

(H_1) φ and ϕ are increasing nonconstant functions defined on $[a, b]$ with $\varphi(a) = \phi(a) = 0$;

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(H_2) f is continuous and there exists $M \geq 0$ such that
$$f : [a,b] \times \Big[-\frac{(b-a)^n(n-1)^{n-1}}{n^n n!}M, +\infty\Big) \to [-M, +\infty);$$
(H_3) h and g are continuous and there exists $M \geq 0$ such that
$$h,\ g : \Big[-\frac{(b-a)^n(n-1)^{n-1}}{n^n n!}M, +\infty\Big) \to [0, +\infty).$$

Lemma 1.1 [1] *Assume that*

(1) $x(t)$ *is a bounded function on* $[a,b]$, *i.e., there exist* c, $C \in \mathbf{R}$ *such that*
$$c \leq x(t) \leq C, \qquad t \in [a,b];$$
(2) $\varphi(t)$ *and* $\phi(t)$ *are increasing on* $[a,b]$;

(3) *Riemann-Stieltjes integrals* $\int_a^b x(t)\mathrm{d}\varphi(\mathrm{t})$ *and* $\int_a^b x(t)\mathrm{d}\phi(\mathrm{t})$ *exist.*

Then there are numbers v_1, $v_2 \in \mathbf{R}$ *with* $c \leq v_1$, $v_2 \leq C$, *such that*
$$\int_a^b x(t)\mathrm{d}\varphi(t) = v_1(\varphi(b) - \varphi(a)),$$
$$\int_a^b x(t)\mathrm{d}\phi(t) = v_2(\phi(b) - \phi(a)).$$

Let $\alpha = \varphi(b)$ and $\beta = \phi(b)$. For any continuous solution $x(t)$ of the BVP (1.1)-(1.2), by Lemma 1.1, there exist ξ, $\eta \in (a,b)$ such that
$$\int_a^b x(t)\mathrm{d}\varphi(t) = x(\xi)(\varphi(b) - \varphi(a)) = x(\xi)\varphi(b) = \alpha x(\xi),$$
$$\int_a^b x(t)\mathrm{d}\phi(t) = x(\eta)(\phi(b) - \phi(a)) = x(\eta)\phi(b) = \beta x(\eta).$$

If
$$h(t) = g(t) = 0, \qquad t \in \Big[-\frac{(b-a)^{n-1}(n-1)^{n-1}}{n^n n!}M,\ 0\Big]$$
and
$$h(t) = g(t) = t, \qquad t \in [0, +\infty),$$
then (1.2) reduces to
$$x(a) = \alpha x(\xi), \quad x'(a) = 0, \quad \cdots, \quad x^{(n-2)}(a) = 0, \quad x(b) = \beta x(\eta). \tag{1.3}$$

The existence of positive solutions of the BVP (1.1)-(1.3) has been studied by several authors when $a = 0$, $b = 1$ and $n = 2$ (see [2–4]).

If
$$h(t) = 0, \qquad t \in \Big[-\frac{(b-a)^{n-1}(n-1)^{n-1}}{n^n n!}M,\ +\infty\Big],$$
$$g(t) = 0, \qquad t \in \Big[-\frac{(b-a)^{n-1}(n-1)^{n-1}}{n^n n!}M,\ 0\Big]$$
and
$$g(t) = t, \qquad t \in [0, +\infty),$$
then (1.2) reduces to
$$x(a) = 0, \quad x'(a) = 0, \quad \cdots, \quad x^{(n-2)}(a) = 0, \quad x(b) = \beta x(\eta). \tag{1.4}$$