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A Kind of Essential Surfaces in the Complements of Knots

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Abstract: In this paper, by the twist-crossing number of knots, we give an upper bound on the Euler characteristic of a kind of essential surfaces in the complements of alternating knots and almost alternating knots, which improves the estimation of the Euler characteristic of the essential surfaces with boundaries under certain conditions. Furthermore, we give the genus of the essential surfaces.

Key words: essential surface, twist-crossing number, almost alternating knot, reduced graph

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1 Introduction

Menasco[1] discussed many properties of incompressible surfaces (including closed incompressible surfaces and essential surfaces) which are properly embedded in the complements of knots, and gave the definition of incompressible pairwise incompressible surfaces. By Menasco's method, if F is a properly embedded essential surface or incompressible pairwise incompressible surface in the complements of alternating knots, we may assume that F is in a standard position, even if F is not in a standard position, F can also be replaced by another surface F', where F' is isotopic to F in the complement of K, and lies in a standard position (see [1–2]). Since surfaces lying in standard position have many good properties, the researchers have used them to analyze properties of surfaces which are properly embedded in the complements of knots. Menasco and Thistlethwaite[2] have proved the cabling conjecture and given an upper bound of the Euler characteristic of a kind of essential surfaces in the complements of alternating knots. Meantime, Menasco also has given a geometric proof to that alternating knots are nontrivial. Adams[3] has dealt with closed incompress-

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ible pairwise incompressible surfaces in the complements of almost alternating knots, and obtained many useful results. Han[4] has studied essential surfaces with meridional boundary in the complements of almost alternating knots, and proved that the essential surfaces with meridional boundary in the complements of almost alternating knots are finite under ambient isotopy. So it is meaningful for us to discuss properties of specific essential surfaces in the complements of alternating knots and almost alternating knots.

In this paper, we mainly discuss properties of a kind of essential surfaces with boundary by using the triangulation of surfaces and the methods adopted by Menasco. The aim of this paper is to give an estimation of the Euler characteristic of the essential surfaces which are properly embedded in the complements of alternating knots and almost alternating knots. In Sections 2 and 3, we give the main theorems of this paper and their corollaries.

2 A Kind of Essential Surfaces in the Complements of Alternating Knots

Definition 2.1 *The twist-crossing number of a diagram D is the smallest positive integer n such that there exists a sequence $a_1, b_1, \cdots, a_n, b_n$, of regular points in cyclic order on the knot K, with the properties that for each $i = 1, 2, \cdots, n$,*

(1) *there is at most one singular point of K between a_i and b_i;*

(2) *all singular points between b_i and a_{i+1} project to crossing points in the same twist of D (a_{n+1} is taken to be a_1).*

Remark 2.1 Denote the twist-crossing number of a diagram D by TCN(D).

Definition 2.2 *Let M be a bounded 3-manifold, and F a properly embedded surface in M. If F is incompressible and boundary incompressible, then F is called an essential surface in M.*

Let $K \subset S^3$ be a knot, and F be a properly embedded surface with boundary in the complement of K and lie in a standard position. Let C be a cell decomposition of F, and $\hat{C}$ be the corresponding cell decomposition of $\hat{F}$, where $\hat{F}$ is the corresponding capped-off closed surface of F. If we take the boundary components of F as fat vertices, then we can get the reduced graph $\hat{C}$ (see [2]).

Lemma 2.1 [4] *Let K be a prime alternating knot, which admits a standard alternating diagram D with $TCN(D) = n$, and let F be an essential surface with finite boundary slope in the exterior of K and with β boundary components, each of which has b ($b \neq 0$) longitudinal components. Then (1) $s = 0$ if $1 \leq n \leq 5$, (2) $s \leq \max\{0, -\chi(F) - b\beta\}$ if $n \geq 6$, where s is the number of saddle-intersections of F with the crossing-balls of D.*

Remark 2.2 When $1 \leq n \leq 5$, all boundary edges and bubbles are good.

Lemma 2.2 [5] *Every closed surface is a polyhedron of some closed and fake 2-manifold.*