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A Class of $*$-simple Type A ω^2-semigroups (I)

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Abstract: In this paper, we study $*$-simple type A ω^2-semigroups in which $\mathcal{D}^* = \widetilde{D}$ and $\mathcal{D}^*|_{E_S} = \mathcal{M}_d$ by the generalized Bruck-Reilly extension and obtain its structure theorem. We also obtain a criterion for isomorphisms of two such semigroups.

Key words: type A semigroup, $*$-simple ω^2-semigroup, generalized Bruck-Reilly extension

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1 Introduction and Preliminaries

Earlier investigations in [1] studied $*$-bisimple type A ω^2-semigroups whose equivalence D^* and $\widetilde{D}$ coincide, characterizing them as the generalized Bruck-Reilly $*$-extensions of cancellative monoids. The results of [1] generalize those of regular bisimple ω^2-semigroups. In this paper, as a natural follow up on these investigations, we study $*$-simple type A ω^2-semigroups in which $\mathcal{D}^* = \widetilde{D}$ and $\mathcal{D}^*|_{E_S} = \mathcal{M}_d$.

The theory developed here closely parallels the one for regular simple ω^2-semigroups. In Sections 2 and 3, it is shown that the $*$-simple type A ω^2-semigroups in which $\mathcal{D}^* = \widetilde{D}$ and $\mathcal{D}^*|_{E_S} = \mathcal{M}_d$ are precisely the generalized Bruck-Reilly extensions of an ω-chain of cancellative monoids of length d. In Section 4, we obtain an isomorphism theorem for such semigroups.

We complete this section with a summary of notions of type A semigroups, the details of which can be found in [1–3].

For any semigroup S we denote by E_S the set of idempotents of S. We define a partial ordering $\geq$ on E_S by the rule that $e \geq f$ if and only if $ef = f = fe$. Let $a, b \in S$ such that for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$. Then a, b are said to be $\mathcal{L}^*$-equivalent and

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written $a\mathcal{L}^*b$. Dually, $a\mathcal{R}^*b$ if for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$. If S has an idempotent e, the following characterisation is known.

Lemma 1.1[3] *Let S be a semigroup, and e be an idempotent in S. Then the following are equivalent*:

(i) $e\mathcal{L}^*a$;

(ii) *$ae = a$ and for all $x, y \in S^1$, $ax = ay$ implies $ex = ey$.*

By duality, a similar condition holds for $\mathcal{R}^*$. A semigroup in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contain an idempotent is called an abundant semigroup (see [2]). The join of the equivalence relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. Thus $a\mathcal{H}^*b$ if and only if $a\mathcal{L}^*b$ and $a\mathcal{R}^*b$. In general, $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$ and neither equals $\mathcal{D}^*$. Basically, $a\mathcal{D}^*b$ if and only if there exist elements x_1, x_2, $\cdots$, x_{2n-1} in S such that $a\mathcal{L}^*x_1\mathcal{R}^*x_2\mathcal{L}^* \cdots \mathcal{L}^*x_{2n-1}\mathcal{R}^*b$. Let H^* be an $\mathcal{H}^*$-class in a semigroup S with $e \in H^*$, where e is an idempotent in S. Then H^* is a cancellative monoid. Denote by $\mathcal{R}$, $\mathcal{L}$ the left and right Green's relations respectively, on S. It is well-known that $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathcal{R} \subseteq \mathcal{R}^*$, $\mathcal{D} \subseteq \mathcal{D}^*$, $\mathcal{H} \subseteq \mathcal{H}^*$ for a semigroup S and if a, b are regular elements of S, then $a\mathcal{L}^*b$ ($a\mathcal{R}^*b$) if and only if $a\mathcal{L}b$ ($a\mathcal{R}b$).

To avoid ambiguity we at times denote a relation $\mathcal{K}$ on S by $\mathcal{K}(S)$. The following notation will be used. An $\mathcal{L}^*$-class containing an element $a \in S$ is denoted by L^*_a. Similarly, R^*_a is an $\mathcal{R}^*$-class with an element $a \in S$. Let S be a semigroup and I an ideal of S. Then I is called a ∗-ideal if $L^*_a \subseteq I$ and $R^*_a \subseteq I$ for all $a \in I$. The smallest ∗-ideal containing a is the principal ∗-ideal generated by a and is denoted by $J^*(a)$. For a, b in S, $a\mathcal{J}^*b$ if and only if $J^*(a) = J^*(b)$. The relation $\mathcal{J}^*$ contains $\mathcal{D}^*$. A semigroup S is said to be ∗-simple if the only ∗-ideal of S is itself. Clearly, a semigroup is ∗-simple if all its elements are $\mathcal{J}^*$-related. Let S be a semigroup with a semilattice E of idempotents. Then S is called a right adequate semigroup if each $\mathcal{L}^*$-class of S contains a unique idempotent. Dually, we have the notion of a left adequate semigroup. A semigroup which is both left and right adequate is called an adequate semigroup. In an adequate semigroup each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contain unique idempotent. For an element x of an adequate semigroup S, $x^*(x^+)$ denotes the unique idempotent in the $\mathcal{L}^*$-class L^*_x ($\mathcal{R}^*$-class R^*_x) of x. A right (left) adequate semigroup S is called a right (left) type A semigroup if $ea = a(ea)^*$ ($ae = (ae)^+a$) for all elements a in S and all idempotents e in S. An adequate semigroups S is type A if it is both right and left type A.

Lemma 1.2[4] *Let S be an arbitrary semigroup. Then the following are equivalent*:

(i) *For all idempotents e and f of S the element ef is regular*;

(ii) *$\langle E_S \rangle$ is a regular subsemigroup*;

(iii) *$Reg(S)$ is a regular subsemigroup.*

Recall that $S = \bigcup\limits_{\alpha \in Y} S_\alpha$ is a strong semilattice of the semigroups S_α when Y is a semilattice, $\{S_\alpha : \alpha \in Y\}$ is a disjoint family of semigroups and for $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there are homomorphisms $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ satisfying