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The First Initial Boundary Value Problem for Parabolic Hessian Equations on Riemannian Manifolds

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Abstract: For a class of elliptic Hessian operators, one type of corresponding parabolic Hessian equations is studied on Riemannian manifolds. The existence and uniqueness of the admissible solution to the first initial boundary value problem for the equations are shown.

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1 Introduction

In this paper, we study the first initial boundary value problem for a class of fully nonlinear parabolic equations of Monge-Ampère type on Riemannian manifolds. Let Ω be a domain of a Riemannian manifold $(M^n,\, g)$ with $n \geq 2$. We consider the following problem:

$$\begin{cases} -D_t u g^{-1}(x) f(\lambda(\nabla^2 u)) = \psi(x,t), & (x,t) \in Q, \\ u(x,t) = \varphi(x,t), & (x,t) \in \partial_p Q, \end{cases} \tag{1.1}$$

where $D_t = \dfrac{\partial}{\partial t}$, $Q = \Omega \times (0, T]$, $\partial_p Q = \partial\Omega \times (0,T] \cup \bar{\Omega} \times \{t = 0\}$, $T > 0$, $\nabla^2 u$ denotes the Hessian matrix of u on M^n, f is a symmetric function with respect to $\lambda \in \mathbf{R}^n$, ψ and φ are functions defined on $\bar{Q}$ and $\partial_p Q$, respectively, and for a $(0,2)$ tensor h on M^n, $\lambda(h) = (\lambda_1, \cdots, \lambda_n)$ denotes the eigenvalues of h with respect to some metric g.

The problem (1.1) with Hessian operators constitute an important class of fully nonlinear equations. The study on Dirichlet problem of (1.1) containing Hessian operator initiated from [1]. It is well known that the classic Monge-Ampère operator is one of Hessian operators. General elliptic Hessian equations have been studied by [2–6]. In 1999, the existence and

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uniqueness of solution to the Dirichlet problem of elliptic equation which contains Hessian operator were established in [3]. For the parabolic case, it seems that the discussion on the equations with Hessian operators initiated from [7]. However, the operators are not of the common form. Ren[8] extended the result of [7], and Wang and Liu[9] studied another type of equations, i.e.,

$$-D_t u + f(\lambda(D^2 u)) = \psi(x,t).$$

In 1996, the Dirichlet problem for elliptic Monge-Ampère equation on Riemannian manifolds was investigated in [10].

Ren and Wang[11] have discussed a type of parabolic Monge-Ampère equations which was introduced by Krylov[12]. In this paper, we generalize our former work and establish the existence and uniqueness of solution to the first initial boundary value problem for parabolic equation which contains Hessian operator.

Throughout this paper, let $f(\lambda)$ be a smooth function defined in an open convex cone $\Gamma \subset \mathbf{R}^n$ with vertex at the origin and containing the positive cone

$$\{\lambda \in \mathbf{R}^n,\ \text{each component } \lambda_i > 0\}.$$

It satisfies that for any i,

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0, \tag{1.2}$$

and

$$f \quad \text{is a concave function.} \tag{1.3}$$

Assume that Γ and f are invariant under interchange of any two λ_i, namely, they are symmetric in λ_i. It follows that $\Gamma \subset \{\sum \lambda_i > 0\}$. For every $C > 0$ and every compact set K in Γ there is a number $R = R(C, K)$ such that

$$f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) \geq C, \qquad \lambda \in K, \tag{1.4}$$

$$f(R\lambda) \geq C, \qquad \lambda \in K. \tag{1.5}$$

In addition, as in [3], assume that f satisfies

$$\sum_{i=1}^{n} f_i(\lambda)\lambda_i \geq 0, \qquad \lambda \in \Gamma, \tag{1.6}$$

$$f_j(\lambda) \geq \nu_0\Big(1 + \sum_i f_i\Big), \qquad \lambda \in \Gamma,\ \lambda_j < 0, \tag{1.7}$$

$$\limsup_{\lambda \to \lambda_0} f(\lambda) \leq 0, \qquad \lambda_0 \in \partial\Gamma, \tag{1.8}$$

and

$$(f_1 \cdots f_n)^{\frac{1}{n}} \geq \mu_0 \qquad \text{in } \{\lambda \in \Gamma : \psi_0 \leq f(\lambda) \leq \psi_1\},\ \psi_1 > \psi_0 > 0, \tag{1.9}$$

where ν_0 and $\mu_0 = \mu_0(\psi_0, \psi_1)$ are some uniform positive constants.

Considering the shape of Ω, we suppose that there exists a sufficiently large constant R such that for every point $x \in \partial\Omega$,

$$(\kappa_1,\ \cdots,\ \kappa_{n-1},\ R) \in \Gamma, \tag{1.10}$$

where $\kappa_1,\ \cdots,\ \kappa_{n-1}$ denote the principal curvatures of $\partial\Omega$ (related to the interior normal).

Let

$$\mathcal{K} = \{v \in C^{2,1}(Q) : \lambda(\nabla^2 v) \in \Gamma,\ -D_t v(x,t) > 0,\ (x,t) \in \bar{Q},\ v = \varphi \text{ on } \partial Q\}.$$