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A Class of Regular Simple ω^2-semigroups-II

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Abstract: In this paper, we study regular simple ω^2-semigroups in which $\mathcal{D}|_{E_S} = \mathcal{W}_d$ by the generalized Bruck-Reilly extension and obtain its structure theorem.

Key words: regular semigroup, simple ω^2-semigroup, generalized Bruck-Reilly extension

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1 Introduction

For any semigroup S we denote by E_S the set of idempotents of S. Let S be a semigroup whose set E_S is non-empty. We define a partial order "$\geq$" on E_S such that $e \geq f$ if and only if $ef = f = fe$. Denote by $\mathbf{N}$ the set of all non-negative integers and by $\mathbf{N}^+$ the set of all positive integers. If $E_S = \{e_i : i \in \mathbf{N}\}$ and the elements of E_S form a chain $e_0 > e_1 > e_2 > \cdots$, then S is called an ω-semigroup. We denote by C_ω the set $\{e_0, e_1, e_2, \cdots\}$ with $e_0 > e_1 > e_2 > \cdots$ We partially order $\mathbf{N} \times \mathbf{N}$ in the following manner: for $(m, n),\ (p, q) \in \mathbf{N} \times \mathbf{N}$,

$$(m, n) \leq (p, q) \text{ if and only if } m > p, \text{ or } m = p \text{ and } n \geq q.$$

The set $\mathbf{N} \times \mathbf{N}$ with the partial order is called an ω^2-chain, and is denoted by C_ω^2. Any partially ordered set order isomorphic to C_ω^2 is also called an ω^2-chain. We say that a semigroup S is an ω^2-semigroup if E_S is order isomorphic to C_ω^2. Thus, if S is an ω^2-semigroup, then we can write

$$E_S = \{e_{m,n} : m, n \in \mathbf{N}\},$$

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where $e_{m,n} \le e_{p,q}$ if and only if $(m,n) \le (p,q)$. In [1], the structure of a bisimple ω-semigroup is shown to be determined entirely by its group of units and an endomorphism of its group of units. Munn[2] studied and classified regular ω-semigroups which are entirely determined by a sequence of groups G_i $(i = 0, 1, \cdots, d-1)$, homomorphism γ_{d-1} of the form $G_{d-1} \to G_0$ and homomorphism γ_i of the form $G_i \to G_{i+1}$ $(i = 0, \cdots, d-2)$ if $d > 1$, where the integer d is characterized as the number of distinct $\mathcal{D}$-classes in this kind of semigroups. Warne[3] investigated the bisimple ω^n-semigroups, and proved that any bisimple ω^n-semigroup has a structure as $(G \times C_n, \circ)$, where G is a group and C_n is a $2n$-cyclic semigroup, under a suitable multiplication.

Regular simple ω^2-semigroups can be regarded as a natural generalization of regular simple ω-semigroups. We manage to do the similar work in [2] in this paper for regular simple ω^2-semigroups. Section 2 presents some information and other necessary notations and terminology. In Section 3 we construct a regular simple ω^2-semigroup in which $\mathcal{D}|_{E_S} = \mathcal{W}_d$ from a sequence of groups G_i $(i = 0, 1, \cdots, d-1)$, homomorphism β, homomorphism γ and an element u of G_0 by the generalized Bruck-Reilly extension. The integer d is characterized as the number of distinct $\mathcal{D}$-classes in such semigroups. It is then proved in Section 4 that this construction provides the most general regular simple ω^2-semigroup in which $\mathcal{D}|_{E_S} = \mathcal{W}_d$.

2 Definitions and Preliminaries

We use the terminologies and notations in Howie[4] and Petrich[5]. Let a, b be the elements of a semigroup S. Then a and b are said to be $\mathcal{L}[\mathcal{R}]$-equivalent if and only if $S^1a = S^1b$ $[aS^1 = bS^1]$. We write $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Then $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ are equivalence relations on S such that $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D}$. Write $\mathcal{D}|_{E_S} = \mathcal{D} \cap (E_S \times E_S)$. A semigroup is called bisimple if it contains only one $\mathcal{D}$-class. A semigroup without zero is called simple if it has no proper ideals. A semigroup is simple if and only if for a, b in S there exist x, y in S such that $xay = b$. We denote by $L_a[R_a, H_a, D_a]$ the $\mathcal{L}[\mathcal{R}, \mathcal{H}, \mathcal{D}]$-class of S containing the element a. For an ω^2-semigroup S with $E_S = \{e_{m,n} : m, n \in \mathbf{N}\}$, let $R_{m,n}$ be the $\mathcal{R}$-class containing an idempotent $e_{m,n}$, and $L_{p,q}$ be the $\mathcal{L}$-class with idempotent $e_{p,q}$, that is,

$$R_{m,n} = \{a \in S : a\mathcal{R}e_{m,n}\}, \qquad L_{p,q} = \{a \in S : a\mathcal{L}e_{p,q}\}.$$

Let $H_{(m,n),(q,p)}$ be the $R_{m,n} \cap L_{p,q}$, that is,

$$H_{(m,n),(q,p)} = \{a \in S : e_{m,n}\mathcal{R}a\mathcal{L}e_{p,q}\}.$$

If $H_{(m,n),(q,p)} \neq \emptyset$, then, evidently, $H_{(m,n),(q,p)}$ is an $\mathcal{H}$-class of S.

Wang and Shang[6] have shown that the set $S = \mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \mathbf{N}$ with the operation defined by

$$(m,\, n,\, q,\, p)(a,\, b,\, d,\, c) = \begin{cases} (m,\ \max\{q, b\} - q + n,\ \max\{q, b\} - b + d,\ c), & p = a; \\ (m,\ n,\ q,\ p - a + c), & p > a; \\ (a - p + m,\ b,\ d,\ c), & p < a \end{cases}$$

is a bisimple ω^2-semigroup and called it the quadrucyclic semigroup, which is denoted by B_{ω^2}. Wang and Shang[6] have introduced the generalized Bruck-Reilly extension also. Let