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Existence of Solutions for a Four-point Boundary Value Problem with a $p(t)$-Laplacian

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Abstract: This paper deals with the existence of solutions to the $p(t)$-Laplacian equation with four-point boundary conditions. It is shown, by Leray-Schauder fixed point theorem and degree method, that under suitable conditions, solutions of the problem exist. The interesting point is that $p(t)$ is a general function.

Key words: $p(t)$-Laplacian, four-point boundary condition, fixed point theorem, Leray-Schauder degree method

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1 Introduction

In this paper, we investigate the existence of solutions to the following $p(t)$-Laplacian ordinary differential equations with four-point boundary conditions:

$$\begin{cases} (|u'(t)|^{p(t)-2}u'(t))' + a(t)f(t,\ u(t),\ u'(t)) = 0, & t \in (0,1), \\ u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0, \end{cases} \tag{1.1}$$

where the functions $f,\ p,\ a$, and the constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfy:

(H1) $f \in C([0,1] \times \mathbf{R} \times \mathbf{R},\ \mathbf{R})$, $p \in C([0,1],\ \mathbf{R})$ and $p(t) > 1$, $a \in C((0,1),\ \mathbf{R})$ is probably singular at $t = 0$ or $t = 1$ and satisfies $0 < \int_0^1 |a(t)|\mathrm{d}t < +\infty$.

(H2) $\alpha,\ \beta > 0$, and $0 < \xi < \eta < 1$.

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In the recent years, differential equations and variational problems with variable exponent have been studied extensively, for which the readers may refer to [1–6]. Such problems arise in the study of image processing, electrorheological fluids dynamics and elastic mechanics (see [7–10]).

In the case when p is a constant, the problem (1.1) becomes the classical p-Laplacian problem. Lian and Ge[11] discussed the following problem:

$$\begin{cases} (|u'(t)|^{p-2}u'(t))' + f(t,\ u(t)) = 0, & 0 < t < 1, \\ u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0, \end{cases}$$

and obtained the existence of multiple positive solutions. For more information about the existence of solutions for ordinary differential equations with p-Laplacian operator, the interested readers may refer to [12–17] and references therein.

Motivated by the results of the above papers, we study the existence of solutions to the problem (1.1). The main features of this paper are as follows. Firstly, $p(t)$ is a general function, which is more complicated than the case when p is constant. Secondly, the nonlinear term f may change sign and $a(t)$ is allowed to be singular at $t = 0$ or $t = 1$, which differ from those p-Laplacian problems.

The outline of this paper is as follows. In Section 2, we give some necessary preliminaries and important lemmas. Sections 3 is devoted to the proof of the existence of solutions to the problem (1.1).

2 Preliminaries

In this section, we give some preliminaries and lemmas.

Define $U = C^1[0, 1]$. It is well known that U is a Banach space with the norm $\|\cdot\|_1$ defined by

$$\|u\|_1 = \|u\| + \|u'\|,$$

where

$$\|u\| = \max_{t\in[0,1]} |u(t)|, \qquad \|u'\| = \max_{t\in[0,1]} |u'(t)|.$$

Set

$$p^- = \min_{t\in[0,1]} p(t), \qquad p^+ = \max_{t\in[0,1]} p(t).$$

Denote

$$\varphi(r, x) = |x|^{p(r)-2}x \qquad \text{for any fixed } r \in [0, 1],\ x \in \mathbf{R},$$

and denote $\varphi^{-1}(r,\ \cdot\ )$ as

$$\varphi^{-1}(r, x) = |x|^{\frac{2-p(r)}{p(r)-1}}x \qquad \text{for any fixed } r \in [0, 1],\ x \in \mathbf{R}\backslash\{0\},$$

where $\varphi^{-1}(r, 0) = 0$.

Evidently, $\varphi^{-1}(r,\ \cdot\ )$ is continuous and sends a bounded sets into a bounded sets.

To obtain the existence of solutions of the problem (1.1), we need the following lemmas. The proofs are standard, and we omit the details.