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On Properties of p-critical Points of Convex Bodies

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Abstract: Properties of the p-measures of asymmetry and the corresponding affine equivariant p-critical points, defined recently by the second author, for convex bodies are discussed in this article. In particular, the continuity of p-critical points with respect to p on $(1, +\infty)$ is confirmed, and the connections between general p-critical points and the Minkowski-critical points (∞-critical points) are investigated. The behavior of p-critical points of convex bodies approximating a convex bodies is studied as well.

Key words: convex body, p-critical point, Minkowski measure of asymmetry, p-measure of asymmetry

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1 Introduction

The measures of asymmetry (or symmetry) for convex bodies initiated from the early work of Minkowski[1] have stayed stably as a quite popular topic in convex geometrical analysis. Many kinds of measure of asymmetry, most of which appear as maximum-minimum problems, have been proposed and studied (cf. [2–12] and the references therein). Even recently some new measures of asymmetry were discovered (see [13–20]).

In [13], The second author introduced a family of measures of (central) asymmetry, called the p-measures of asymmetry, which have the well-known Minkowski measure as a special case, and showed some similar properties of the p-measures to the well-known Minkowski one. However, even if it was proved in [13] that for $1 < p < +\infty$, the p-critical points of a

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convex body are unique, the general behavior of p-measures as p varying is still not clear. So, in this paper, we discuss the continuity of p-critical points with respect to p and the connections between p-critical points ($1 < p < +\infty$) and the Minkowski-critical points (∞-critical points). The behavior of p-critical points of convex bodies approximating a convex bodies is studied as well.

2 Preliminaries

$\mathbf{R}^n$ denotes the usual n-dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$. A bounded closed convex set $C \subset \mathbf{R}^n$ is called a convex body if it has non-empty interior (int for brief). The family of all convex bodies is denoted by $\mathcal{K}^n$. Other notation and terms are referred to [21].

For $C \in \mathcal{K}^n$, denoted by $h(C, u)$, $u \in \mathbb{S}^{n-1}$, the support function of C, where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere. For $x \in \mathbf{R}^n$, we denote also

$$h_x(C, u) = \sup_{y \in C} \langle y - x, u \rangle, \qquad u \in \mathbb{S}^{n-1},$$

which is called the support function of C based at x. Clearly we have

$$h_x(C, \cdot) = h(C, \cdot) - \langle x, \cdot \rangle.$$

Furthermore, if, for $x \in \mathbf{R}^n$, writing $C_x := C - x$, we also have

$$h_x(C, \cdot) = h(C_x, \cdot).$$

The Hausdorff metric $d_H(C, D)$ between $C, D \in \mathcal{K}^n$ is defined as

$$d_H(C, D) := \max_{u \in \mathbb{S}^{n-1}} |h(C, u) - h(D, u)|.$$

We recall now the definition of p-measures of asymmetry (see [13]).

Given $C \in \mathcal{K}^n$, for a fixed $x \in \text{int}(C)$, we define a probability measure $m_x(C, \cdot)$ on $\mathbb{S}^{n-1}$ by

$$m_x(C, \omega) := \frac{\displaystyle\int_\omega h_x(C, u) \mathrm{d}S_{n-1}(C, u)}{nV_n(C)} \qquad \text{for any measurable } \omega \subset \mathbb{S}^{n-1},$$

where $S_{n-1}(C, \cdot)$ denotes the surface area measure of C on $\mathbb{S}^{n-1}$ and $V_n(C)$ denotes the n-dimensional volume of C. Then write

$$\mu_p(C, x) := \begin{cases} \left(\displaystyle\int_{\mathbb{S}^{n-1}} \alpha_x(C, u)^p \mathrm{d}m_x(C, u) \right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty; \\ \displaystyle\sup_{u \in \mathbb{S}^{n-1}} \alpha_x(C, u), & \text{if } p = \infty, \end{cases}$$

where

$$\alpha_x(C, u) = \frac{h(C_x, -u)}{h(C_x, u)}.$$

Definition 2.1[13] *For $C \in \mathcal{K}^n$ and $1 \le p \le \infty$, the p-measure of asymmetry* $\text{as}_p(C)$ *of C is defined by*

$$\text{as}_p(C) := \inf_{x \in \text{int}(C)} \mu_p(C, x).$$

A point $x \in \text{int}(C)$ satisfying $\mu_p(C, x) = \text{as}_p(C)$ is called a p-critical point of C.