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# Lipschitz Estimates for Commutators of $p$-adic Fractional Hardy Operators

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**Abstract:** In this paper, the boundedness of commutators generated by the $n$-dimensional fractional Hardy operators and Lipschitz functions on $p$-adic function spaces are obtained. The authors show that these commutators are bounded on Herz space and Lebesgue space with suitable indexes. Moreover, the commutator of Hardy-Littlewood-Polyá operator is also considered.

**Key words:** commutator, the $p$-adic Hardy operator, Lipschitz function, Herz space

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## 1 Introduction

Let $f$ be a locally integrable function on $\mathbf{R}^n$. The famous Hardy operators are defined by

$$\mathcal{H}f(x)=\frac{1}{|x|^n}\int_{|t|<|x|}f(t)\mathrm{d}t,\quad \mathcal{H}^*f(x)=\int_{|t|\geq|x|}\frac{f(t)}{t^n}\mathrm{d}t,\qquad x\in\mathbf{R}^n\backslash\{0\}.$$

They are adjoint mutually and satisfy

$$\int_{\mathbf{R}^n}g(x)\mathcal{H}f(x)\mathrm{d}x=\int_{\mathbf{R}^n}f(x)\mathcal{H}^*g(x)\mathrm{d}x,$$

where $f\in L^p(\mathbf{R}^n)$, $g\in L^{p'}(\mathbf{R}^n)$ for $1<p<\infty$ and $\dfrac{1}{p}+\dfrac{1}{p'}=1$.

The analysis of Hardy-type operators have drawn considerable attentions and a lot of works about the boundedness of commutators of various Hardy-type operators on Euclidean spaces. The most celebrated Hardy's integral inequality was established by Hardy[1], where he proved the $(L^p(\mathbf{R}^+),\ L^{p'}(\mathbf{R}^+))$ boundedness for $\mathcal{H}$ and $\mathcal{H}^*$, respectively. Moreover, he

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$$\|\mathcal{H}\|_{L^p\to L^p} = \|\mathcal{H}^*\|_{L^{p'}\to L^{p'}} = \frac{p}{p-1}$$

by giving a simple proof of Hilbert's double series theorem. Christ and Grafakos[2] proved the same bound for $\mathcal{H}$ on higher dimensional spaces.

As a systematic works on Hardy-type operators and their commutators, Fu *et al.*[3] gave a new characterization of CMO($\mathbf{R}^n$) (central BMO($\mathbf{R}^n$) space) via the commutator of $n$-dimensional fractional Hardy operator which was defined in [3] as follows:

$$\mathcal{H}_\beta f(x) = \frac{1}{|x|^{n-\beta}}\int_{|t|<|x|} f(t)\mathrm{d}t,$$

$$\mathcal{H}^*_\beta f(x) = \int_{|t|\geq|x|} \frac{f(t)}{t^{n-\beta}}\mathrm{d}t, \qquad x\in \mathbf{R}^n\backslash\{0\},\ \beta\in[0,n).$$

Fu and Lu[4] introduced the commutators generated by the generalized Hardy operators and CMO($\mathbf{R}^n$) functions, and considered the boundedness of these commutators on homogeneous Herz spaces. The endpoint estimates for commutators were also obtained in [5]. On the other hand, Zheng and Fu[6] obtained the boundedness of commutators generated by Hardy operators and Lipschitz functions on Lebesgue and Herz spaces.

Recently, the harmonic analysis on $p$-adic field has been drawing more and more concern, such as [7–8] about Riesz potentials, [9–11] about $p$-adic pseudodifferential equations and wavelet in [12].

Firstly, we introduce some basic and necessary notations for the $p$-adic field.

For a prime number $p$, $Q_p$ denotes the field of $p$-adic numbers. This field is the completion of the field of rational numbers $Q$ with respect to the non-Archimedean $p$-adic norm $|\cdot|_p$, which is defined as follows:

(i) $|0|_p = 0$;

(ii) if any non-zero rational number $x$ is represented as $x = p^r\frac{m}{n}$, where $m$ and $n$ are integers which are not divisible by $p$, and $r = r(x)$ is an integer, then $|x|_p = p^{-r}$.

It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p|y|_p;$$

$$|x+y|_p \leq \max\{|x|_p,\ |y|_p\}.$$

From the second property, we know that $|x+y|_p = \max\{|x|_p,\ |y|_p\}$ if and only if $|x|_p \neq |y|_p$.

We define the space

$$Q_p^n = \{x=(x_1,x_2,\cdots,x_n),\ x_i\in Q_p,\ i=1,2,\cdots n\}$$

with the norm

$$|x|_p = \max\{|x_i|_p,\ i=1,\cdots,n\}.$$

We denote by

$$B_r(a) = \{x\in Q_p^n : |x-a|\leq p^r\},$$

the ball with center at $a\in Q_p^n$ and radius $p^r$, and

$$S_r(a) = \{x\in Q_p^n : |x-a|_p = p^r\} = B_r(a)\backslash B_{r-1}(a).$$