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Boundedness of Commutators Generated by Campanato-type Functions and Riesz Transforms Associated with Schrödinger Operators

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Abstract: Let $\mathfrak{L} = -\Delta + V$ be a Schrödinger operator on $\mathbf{R}^n$, $n > 3$, where Δ is the Laplacian on $\mathbf{R}^n$ and $V \neq 0$ is a nonnegative function satisfying the reverse Hölder's inequality. Let $[b, T]$ be the commutator generated by the Campanato-type function $b \in \Lambda_{\mathfrak{L}}^{\beta}$ and the Riesz transform associated with Schrödinger operator $T = \nabla(-\Delta+V)^{-\frac{1}{2}}$. In the paper, we establish the boundedness of $[b, T]$ on Lebesgue spaces and Campanato-type spaces.

Key words: commutator, Campanato-type space, Riesz transform, Schrödinger operator

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1 Introduction

Let $\mathfrak{L} = -\Delta + V$ be a Schrödinger operator on $\mathbf{R}^n$, $n > 3$, where Δ is the Laplacian on $\mathbf{R}^n$ and $V \neq 0$ is a nonnegative locally integrable function. The problems related to the Schrödinger operators $\mathfrak{L}$ have attracted much attention (see [1–3] for example). In particular, Fefferman[1], Shen[2] and Zhong[3] established some basic results about the fundamental solutions and the boundedness of Riesz transforms associated with the Schrödinger operator.

The commutators generated by the Riesz transform associated with Schrödinger operator and BMO functions or Lipschitz functions also attract much attention (see [4–7] for example). Chu[8] considered the boudedness of commutators generalized by the $\mathrm{BMO}_{\mathfrak{L}}$ function

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and the Riesz transform $\nabla(-\Delta+V)^{-\frac{1}{2}}$ on Lebesgue spaces. And Jiang[9] investigates some properties of the Riesz potential $(-\Delta+V)^{-\frac{\alpha}{2}}$ on the Campanato-type spaces $\Lambda_{\mathfrak{L}}^{\beta}$. Inspired by [4, 6, 8–9], in this paper we consider the boundedness of commutators generated by the Campanato-type function $b\in\Lambda_{\mathfrak{L}}^{\beta}$ and the Riesz transform $\nabla(-\Delta+V)^{-\frac{1}{2}}$ on Lebesgue spaces and Campanato-type spaces.

Firstly, let us introduce some notations. A nonnegative locally $L^q(\mathbf{R}^n)$ integrable function V is said to belong to B_q $(1<q<\infty)$ if there exists a constant $C=C(q, V)>0$ such that the reverse Hölder's inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q\mathrm{d}x\right)^{\frac{1}{q}}\le C\left(\frac{1}{|B|}\int_B V(x)\mathrm{d}x\right) \tag{1.1}$$

holds for any ball B in $\mathbf{R}^n$.

We also say a nonnegative function $V\in B_\infty$, if there exists a constant $C>0$ such that

$$\max_{x\in B}V(x)\le C\left(\frac{1}{|B|}\int_B V(x)\mathrm{d}x\right)$$

holds for any ball B in $\mathbf{R}^n$.

By Hölder's inequality, we have $B_{q_1}\subset B_{q_2}$ for $q_1>q_2>1$. One remarkable feature about the B_q class is that if $V\in B_q$ for some $q>1$, then there exists an $\varepsilon>0$ which depends only on n and the constant C in (1.1) such that $V\in B_{q+\varepsilon}$. It is also well known that if $V\in B_q$ $(q>1)$, then $V(x)\mathrm{d}x$ is a doubling measure, namely, for any $r>0$, $x\in\mathbf{R}^n$ and some constant C_0, we have

$$\int_{B(x,2r)}V(y)\mathrm{d}y\le C_0\int_{B(x,r)}V(y)\mathrm{d}y.$$

Definition 1.1[3] *For $x\in\mathbf{R}^n$, the function $m(x,V)$ is defined by*

$$\frac{1}{m(x,V)}=\sup\left\{r>0:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)\mathrm{d}y\le 1\right\}.$$

Clearly, $0<m(x,V)<1$ for every $x\in\mathbf{R}^n$ and if $r=m(x,V)$, then

$$\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)\mathrm{d}y=1.$$

For simplicity, we denote $\frac{1}{m(x,V)}$ by $\rho(x)$.

Definition 1.2[9–11] *Let $\mathfrak{L}=-\Delta+V$, $p\in(0,\infty)$ and $\beta\in\mathbf{R}^n$. A function $f\in L^p_{\mathrm{loc}}(\mathbf{R}^n)$ is said to be in $\Lambda_{\mathfrak{L}}^{\beta,p}(\mathbf{R}^n)$, if there exists a nonnegative constant C such that for all $x\in\mathbf{R}^n$ and $0<s<\rho(x)\le r$,*

$$\left\{\frac{1}{|B(x,s)|^{1+p\beta}}\int_{B(x,s)}|f(y)-f_{B(x,s)}|^p\mathrm{d}y\right\}^{\frac{1}{p}}+\left\{\frac{1}{|B(x,r)|^{1+p\beta}}\int_{B(x,r)}|f(y)|^p\mathrm{d}y\right\}^{\frac{1}{p}}\le C,$$

where $f_B=\frac{1}{|B|}\int_B f(y)\mathrm{d}y$ for any ball B. Moreover, the minimal constant C as above is defined for the norm of f in the space $\Lambda_{\mathfrak{L}}^{\beta,p}(\mathbf{R}^n)$ and denote by $\|f\|_{\Lambda_{\mathfrak{L}}^{\beta,p}(\mathbf{R}^n)}$.

Remark 1.1 When $p\in[1,\infty)$, $\Lambda_{\mathfrak{L}}^{0,p}(\mathbf{R}^n)=\mathrm{BMO}_{\mathfrak{L}}(\mathbf{R}^n)$. And when $0\le\beta<\infty$ and $p_1,p_2\in[1,\infty)$, $\Lambda_{\mathfrak{L}}^{\beta,p_1}(\mathbf{R}^n)=\Lambda_{\mathfrak{L}}^{\beta,p_2}(\mathbf{R}^n)$ and $\|f\|_{\Lambda_{\mathfrak{L}}^{\beta,p_1}(\mathbf{R}^n)}\sim\|f\|_{\Lambda_{\mathfrak{L}}^{\beta,p_2}(\mathbf{R}^n)}$. For simplicity, we denote $\Lambda_{\mathfrak{L}}^{\beta,p}(\mathbf{R}^n)$ by $\Lambda_{\mathfrak{L}}^{\beta}(\mathbf{R}^n)$.