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Existence of Multiple Positive Periodic Solutions for Second Order Differential Equations

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Abstract: In this paper, we study the existence of multiple positive periodic solutions for the second order differential equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = f(t, x(t)).$$

By using Krasnoselskii fixed point theorem, we establish some criteria for the existence and multiple positive periodic solutions for this differential equation.

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1 Introduction

This paper is concerned with the existence and multiplicity of positive solutions for the following second order differential equation

$$x''(t) + p(t)x' + q(t)x(t) = f(t, x(t)), \qquad t \in \mathbf{R}, \tag{1.1}$$

where $p(t), q(t) \in C(\mathbf{R},\ \mathbf{R}^+)$ are ω-periodic, $f = (f_1,\ f_2,\ \cdots, f_n)^{\mathrm{T}}$, $f(t,\ x(t))$ is a functional defined on $\mathbf{R} \times BC$ and $f(t,\ x(t))$ is ω-periodic in t, BC denotes the Banach space of bounded continuous functions $\phi : \mathbf{R} \to \mathbf{R}^n$ with the norm $\|\phi\| = \sup\limits_{\theta\in\mathbf{R}} \sum\limits_{i=1}^{n} |\phi_i(\theta)|$, where $\phi = (\phi_1,\ \phi_2,\ \cdots,\ \phi_n)^{\mathrm{T}}$.

The existence of positive periodic solutions for differential equations were extensively studied (see [1]–[5]). One of the effective approaches to fulfill such a problem is employing fixed point theorem, and some prior estimations of possible periodic solutions are obtained.

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Raffoul[6] considered the nonlinear neutral differential equation. Liu and Ge[7] investigated the nonlinear Duffing equation. Jiang *et al.*[8] considered the existence and nonexistence of positive periodic solutions to a system of infinite delay equation. Recently, many scholars study second order differential equation and obtain some new results (see [9]–[11]). Motivated by the papers mentioned above, we aim to study the existence of multiple positive periodic solutions for the second order differential equation (1.1).

The following definition and theorem are needed in our arguments.

Definition 1.1 *Let X be a Banach space and E be a closed, nonempty subset of X. E is said to be a cone if*

(i) $\alpha u + \beta v \in E$ *for all* $u,\ v \in E$ *and all* $\alpha,\ \beta > 0$;

(ii) $u,\ -u \in E$ *imply* $u = 0$.

Theorem 1.1 ([12], Krasnoselskii fixed point theorem) *Let X be a Banach space, and E be a cone in X. Suppose that Ω_1 and Ω_2 are open subset of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. Suppose that $T : E \cap (\bar{\Omega}_2 \setminus \Omega_1) \to E$ is a completely continuous operator and satisfies either*

(i) $\|Tx\| \geq \|x\|$ *for any* $x \in E \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|$ *for any* $x \in E \cap \partial\Omega_2$; *or*

(ii) $\|Tx\| \leq \|x\|$ *for any* $x \in E \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|$ *for any* $x \in E \cap \partial\Omega_2$.

Then T has a fixed point in $E \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

This paper is organized as follows. In Section 2, we introduce some notations and preliminary results. Section 3 presents some existence results of positive periodic solutions for (1.1). The multiplicity results are given in Section 4.

2 Some Lemmas

In this section, we make some preparations for our main results.

Let $X = \{x \in C(\mathbf{R},\ \mathbf{R}^n);\ x(t+\omega) = x(t),\ t \in \mathbf{R}\}$ with the norm $\|x\| = \sup\limits_{t\in\mathbf{R}} \sum\limits_{i=1}^{n} |x_i(t)|$. Obviously, X is a Banach space.

Lemma 2.1[7] *Suppose that*

$$\frac{R_1\left[\exp\left\{\displaystyle\int_0^\omega p(u)\mathrm{d}u\right\} - 1\right]}{Q_1\omega} \geq 1,$$

where

$$R_1 = \max_{t\in[0,\omega]} \left| \int_t^{t+\omega} \frac{\exp\left\{\displaystyle\int_t^s p(u)\mathrm{d}u\right\}}{\exp\left\{\displaystyle\int_0^\omega p(u)\mathrm{d}u\right\} - 1} q(s)\mathrm{d}s \right|,$$

$$Q_1 = \left(1 + \exp\left\{\int_0^\omega p(u)\mathrm{d}u\right\}\right)^2 R_1^2.$$

Then there are continuous ω-periodic functions a and b such that

$$b(t) > 0, \quad \int_0^\omega a(u)\mathrm{d}u > 0, \quad a(t) + b(t) = p(t), \quad b'(t) + a(t)b(t) = q(t), \qquad t \in \mathbf{R}.$$