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Multiple Solutions for the Eigenvalue Problem of Nonlinear Fractional Differential Equations

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Abstract: In this article, multiple solutions for the eigenvalue problem of nonlinear fractional differential equation is considered. We obtain the existence and multiplicity results of positive solutions by using some fixed point theorems.

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1 Introduction

Fractional derivative is as old as calculus. In 1695, L'Hospital presented the question: what is the meaning of $\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n}$ if $n = \dfrac{1}{2}$? Since then, many researchers tried to study fractional derivatives and have obtained a large amount of results. Fractional differential and integral equations play increasingly important roles in modeling of engineering and science problems, as shown in [1]–[5]. In many situations, these models provide more suitable results than analogous models with integer derivatives, see [6] for details. In most of the available literatures (see [7]–[13]), the existence results for fractional differential equations obtained by use of fixed point theorem.

In this paper, we investigate the existence and multiplicity of positive solutions for eigenvalue problem of nonlinear fractional differential equations:

$$ {}^{C}D^{\alpha}_{0_+} u(t) + \lambda a(t) f(t, u(t)) = 0, \qquad 0 < t < 1, \tag{1.1}$$

$$ u(0) = u'(1) = u''(0) = 0, \tag{1.2}$$

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where $2 < \alpha \le 3$ is a real number, ${}^C D^{\alpha}_{0_+}$ is the Caputo derivative, λ is a positive parameter and $a(t) \in C([0, 1],\ [0, \infty))$, $f(t, u) \in C([0, 1] \times [0, \infty),\ [0, \infty))$. Firstly, we establish intervals of the parameter λ, which yield the conclusion that the problem (1.1)-(1.2) has a positive solution. Then we give some conditions about $f(t, u)$, which also yield the existence of positive solutions using of fixed point theorem. At the end, by placing certain restrictions on the nonlinearity, we prove the existence of at least one, at least two, at least three, and infinitely many positive solutions of the problem (1.1)-(1.2) by applying some known fixed point theorems.

2 Preliminaries

Definition 2.1 *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbf{R}$ is given by*

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s$$

provided that the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 *The Caputo's derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbf{R}$ is given by*

$${}^C D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s,$$

where $n - 1 < \alpha \le n$, provided that the right hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1[10] *Given $f \in C([0, 1])$, and $2 < \alpha \le 3$, the unique solution of*

$$\begin{aligned} &{}^C D^{\alpha}_{0+} u(t) + f(t) = 0, \qquad 0 < t < 1, \\ &u(0) = u'(1) = u''(0) = 0 \end{aligned}$$

is

$$u(t) = \int_0^1 G(t, s) f(s)\mathrm{d}s,$$

where

$$G(t, s) = \begin{cases} \dfrac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases}$$

Proposition 2.1 *The Green's function $G(t, s)$ satisfies the following conditions*:

(1) $0 < G(t, s) \le G(1, s), \quad t, s \in (0, 1)$;

(2) $\displaystyle\min_{\frac{1}{4} \le t \le \frac{3}{4}} G(t, s) \ge \frac{1}{4} \max_{0 \le t \le 1} G(t, s) = \frac{1}{4} G(1, s), \qquad 0 < s < 1.$

Lemma 2.2[11] *Let X be a Banach space, $K \subseteq X$ is a cone, and $\Omega_1, \Omega_2 \subset K$ are two relatively non-empty open sets, $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. Suppose that $F : \Omega_2 \longrightarrow K$ is a completely continuous operator such that either*

(1) $\|F(x)\| \le \|x\|$, $x \in \partial\Omega_1$; $\|F(x)\| \ge \|x\|$, $x \in \partial\Omega_2$, *or*