Commun. Math. Res.
doi: 10.4208/cmr.2020-0037

Vol. **37**, No. 1, pp. 113-140
February 2021

Gradient Flow of the L_β-Functional

Xiaoli Han[1], Jiayu Li[2,*] and Jun Sun[3,4]

[1] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.*
[2] *School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026 AMSS CAS, Beijing 100190, China.*
[3] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.*
[4] *Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China.*

Received 27 September 2020; Accepted 27 November 2020

Abstract. In this paper, we start to study the gradient flow of the functional L_β introduced by Han-Li-Sun in [8]. As a first step, we show that if the initial surface is symplectic in a Kähler surface, then the symplectic property is preserved along the gradient flow. Then we show that the singularity of the flow is characterized by the maximal norm of the second fundamental form. When $\beta=1$, we derive a monotonicity formula for the flow. As applications, we show that the λ-tangent cone of the flow consists of the finite flat planes.

AMS subject classifications: 53C42, 53C44

Key words: β-symplectic critical surfaces, gradient flow, monotonicity formula, tangent cone.

---

1 Introduction

Suppose that M is a Kähler surface. Let ω be the Kähler form on M and let J be a complex structure compatible with ω. The Riemannian metric $\langle\cdot,\cdot\rangle$ on M is defined by

$$\langle U,V\rangle=\omega(U,JV).$$

---

*Corresponding author. *Email addresses:* hanxiaoli@tsinghua.edu.cn (X. Han), jiayuli@ustc.edu.cn (J. Li), sunjun@whu.edu.cn (J. Sun)

For a compact oriented real surface Σ which is smoothly immersed in M, one defines, following [5], the Kähler angle α of Σ in M by

$$\omega|_\Sigma = \cos\alpha d\mu_\Sigma, \tag{1.1}$$

where $d\mu_\Sigma$ is the area element of Σ of the induced metric from $\langle\cdot,\cdot\rangle$. We say that Σ is a holomorphic curve if $\cos\alpha\equiv 1$, Σ is a Lagrangian surface if $\cos\alpha\equiv 0$ and Σ is a symplectic surface if $\cos\alpha>0$.

The existence of holomorphic curves in a Kähler surface is a fundamental problem in differential geometry. Since holomorphic curves are always area-minimizing in its homological class due to the Wirtinger inequality, we see that holomorphic curves are all stable symplectic minimal surfaces. Wolfson [18] showed that a symplectic minimal surface in a Kähler-Einstein surface with non-negative scalar curvature must be holomorphic. Thus, we can look for the holomorphic curves by finding the symplectic minimal surfaces in this case.

Furthermore, Chen-Li [3] and Wang [17] showed that symplectic property is preserved along the mean curvature flow. Therefore, an idea approaching the existence of holomorphic curves is to looking for symplectic minimal surfaces using the mean curvature flow starting from a symplectic surface, which we call symplectic mean curvature flow. There are some interesting results on the study of symplectic mean curvature flow. For instance, Chen-Li [3] and Wang [17] showed that there is no Type I singularities for such a flow at the finite time. However, since the flow is of codimension two and the normal bundle is much more complex, it is hard to clear out all singularities. On the other hand, C. Arezzo [2] constructed examples which shows that a strictly stable minimal surface in a Kähler-Einstein surface with negative scalar curvature may not be holomorphic.

For this reason, we introduce a new idea to approach the existence of holomorphic curves using variational method combined with the continuity method. More precisely, we consider a sequence of functionals [8]

$$L_\beta = \int_\Sigma \frac{1}{\cos^\beta\alpha} d\mu, \tag{1.2}$$

where $\beta\geq 0$. The functional L_1 was introduce by Han-Li in [7]. The critical point of the functionals L_β in the class of symplectic surfaces in a Kähler surface is called a β-symplectic critical surface. We have proved that (cf., [8]) the Euler-Lagrange equation of the functional L_β is

$$\cos^3\alpha\mathbf{H} - \beta\big(J(J\nabla\cos\alpha)^\top\big)^\perp = 0, \tag{1.3}$$

where $\mathbf{H}$ is the mean curvature vector of Σ in M, and $(\cdot)^\top$ means tangential components of $(\cdot)$, $(\cdot)^\perp$ means the normal components of $(\cdot)$. It is clear that holomorphic curves are β-symplectic critical surfaces for each β. When $\beta=0$, the