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The Core-EP, Weighted Core-EP Inverse of Matrices and Constrained Systems of Linear Equations

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Abstract. We study the constrained system of linear equations

$$Ax=b, \quad x\in\mathcal{R}\left(A^k\right)$$

for $A\in\mathbb{C}^{n\times n}$ and $b\in\mathbb{C}^n, k=\operatorname{Ind}(A)$. When the system is consistent, it is well known that it has a unique A^Db. If the system is inconsistent, then we seek for the least squares solution of the problem and consider

$$\min_{x\in\mathcal{R}\left(A^k\right)} \|b-Ax\|_2,$$

where $\|\cdot\|_2$ is the 2-norm. For the inconsistent system with a matrix A of index one, it was proved recently that the solution is $A^{\#}b$ using the core inverse $A^{\#}$ of A. For matrices of an arbitrary index and an arbitrary b, we show that the solution of the constrained system can be expressed as $A^{\dagger}b$ where $A^{\dagger}$ is the core-EP inverse of A. We establish two Cramer's rules for the inconsistent constrained least squares solution and develop several explicit expressions for the core-EP inverse of matrices of an arbitrary index. Using these expressions, two Cramer's rules and one Gaussian elimination method for computing the

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core-EP inverse of matrices of an arbitrary index are proposed in this paper. We also consider the W-weighted core-EP inverse of a rectangular matrix and apply the weighted core-EP inverse to a more general constrained system of linear equations.

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Key words: Bott-Duffin inverse, Core-EP inverse, weighted core-EP inverse, Cramer's rule, Gaussian elimination method.

1 Introduction

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ matrices over $\mathbb{C}$. For a matrix $A\in\mathbb{C}^{m\times n}$, A^T,A^*,$\mathcal{R}(A)$,$\mathcal{N}(A)$, and $\mathrm{Ind}(A)$ stand for its transpose, conjugate transpose, range, null space, and index. I is the identity matrix of order n and e_i is the i-th column of I. The Moore-Penrose inverse $A^\dagger$ of A is the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying

$$AXA=A, \tag{1.1}$$
$$XAX=X, \tag{1.2}$$
$$(AX)^*=AX, \tag{1.3}$$
$$(XA)^*=XA. \tag{1.4}$$

The matrix X satisfying the 1st and 3rd matrix equations of the system of matrix equations (1.1)-(1.4) is called a $\{1,3\}$-inverse of A, denoted by $A^{(1,3)}$ and the collection of all $\{1,3\}$-inverses of A is denoted by $A\{1,3\}$. It is well known that $A^\dagger=A^{-1}$ for a nonsingular square matrix A and that $A^\dagger b$ is the minimum norm least squares solution of the system of linear equations $Ax=b$ for a general matrix $A\in\mathbb{C}^{m\times n}$ and $b\in\mathbb{C}^m$.

The Drazin inverse A^D of a square matrix $A\in\mathbb{C}^{n\times n}$ is the unique matrix $X\in\mathbb{C}^{n\times n}$ satisfying

$$XAX=X, \quad XA^{k+1}=A^k, \quad AX=XA \tag{1.5}$$

for $k=\mathrm{Ind}(A)$. It is well known that both $A^\dagger$ and A^D coincide with A^{-1} for nonsingular matrices. For the special case when $\mathrm{Ind}(A)$ is one, the Drazin inverse is called the group inverse and is denoted by $A^\#$. The group inverse is a useful tool in the study of Markov chains and the Drazin inverse is used to study the singular differential and difference equations [7]. It is well known that the constrained system of linear equations

$$Ax=b, \quad x\in\mathcal{R}(A^k) \tag{1.6}$$