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Convergence Analysis of Asymptotical Regularization and Runge-Kutta Integrators for Linear Inverse Problems under Variational Source Conditions

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Abstract. Variational source conditions are known to be a versatile tool for establishing error bounds, and these recently attract much attention. We establish sufficient conditions for general spectral regularization methods which yield convergence rates under variational source conditions. Specifically, we revisit the asymptotical regularization, Runge-Kutta integrators, and verify that these methods satisfy the proposed conditions. Numerical examples confirm the theoretical findings.

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1 Introduction

Inverse problems arise in many contexts and have important applications in science and engineering, i.e. [7,19,25]. Their typical feature is the intrinsic ill-posedness which yields unstable numerical reconstruction unless regularization is used. The theory of regularization methods has been well-established in the last decades in solving the inverse problems stably. In this paper we study ill-posed operator equations acting between real Hilbert spaces X and Y. More precisely, denote $A\colon X\to Y$ an injective and bounded linear operator with a non-closed range $\mathcal{R}(A)$, and $x^\dagger$ be the unknown exact solution. We are

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interested in the recovery of the unknown solution by the indirect noisy observation data y^δ, i.e.

$$y^\delta = Ax^\dagger + \xi, \qquad \|\xi\|_Y \le \delta, \tag{1.1}$$

given a noise level $\delta > 0$.

To obtain a stable reconstructed solution of (1.1), regularization methods are necessary and are widely used in real applications. Many of the regularization methods can be seen from two categories: variational regularization methods and spectral regularization methods. For instance, the well-known Tikhonov regularization

$$x_\alpha^\delta \to \operatorname{argmin}_{x\in X}\left\{\|Ax - y^\delta\|_Y^2 + \alpha\|x - x_0\|_X^2\right\},$$

with a regularization parameter $\alpha > 0$ and an initial guess x_0, belongs to the first category. In Hilbert spaces the above minimizer x_α^δ can be given explicitly, and it can be written in a spectral regularization form as $x_\alpha^\delta = x_0 + (\alpha I + A^*A)^{-1} A^*\left(y^\delta - Ax_0\right)$. More generally, we consider spectral regularization methods of the form

$$x_\alpha^\delta = x_0 + g_\alpha(A^*A)A^*(y^\delta - Ax_0), \quad \alpha > 0,$$

where x_0 is the initial guess, and for different filter functions g_α to be defined in Definition 2.1. Generally $g_\alpha(A^*A)$ is a bounded self-adjoint operator on X by the spectral calculus and we refer to [7, Sec. 2.3] for further details.

The performance of regularization depends on *smoothness assumptions*, see e.g. the standard monograph [7], but also [19]. For spectral regularization the most commonly used are source conditions, see [20], which are in the general form of

$$x^\dagger - x_0 \in \mathcal{A}_\psi := \{x,\ x = \psi(A^*A)v,\ \|v\|_X \le 1\} \tag{1.2}$$

with an initial guess x_0. Here ψ is *an index function* and it is a non-decreasing and continuous function $\psi:(0,\infty)\to(0,\infty)$ with $\lim_{\lambda\searrow 0}\psi(\lambda) = 0$. We note that the index function is closely related to the stability property of the inverse problem, and thus the order of the convergence rate, i.e. [5, 6, 18].

However, for variational regularization, in particular for Tikhonov regularization of nonlinear ill-posed problems in Banach spaces with a general penalty functional, starting from [13], it has proven useful to consider *the variational source conditions* (VSC). We refer to [14, 24, 25] for extended discussion. These variational source conditions can be formulated by

$$2\langle x^\dagger - x_0, x^\dagger - x\rangle \le \frac{1}{2}\|x - x^\dagger\|_X^2 + \varphi\left(\|Ax - Ax^\dagger\|_Y\right), \quad \forall x \in X. \tag{1.3}$$

Again, φ in (1.3) is supposed to be an index function as described above. One advantage of VSC consists in its immediate extension to operator equations in Banach spaces, and