J. Math. Study
doi: 10.4208/jms.v53n4.20.01

Vol. **53**, No. 4, pp. 370-401
December 2020

ODE Methods in Non-Local Equations

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Received October 31, 2019; Accepted March 3, 2020;
Published online December 22, 2020.

Dedicated to Professors Sun-Yung Alice Chang and Paul C.-P. Yang on their 70th birthdays

Abstract. Non-local equations cannot be treated using classical ODE theorems. Nevertheless, several new methods have been introduced in the non-local gluing scheme of our previous article; we survey and improve those, and present new applications as well. First, from the explicit symbol of the conformal fractional Laplacian, a variation of constants formula is obtained for fractional Hardy operators. We thus develop, in addition to a suitable extension in the spirit of Caffarelli–Silvestre, an equivalent formulation as an infinite system of second order constant coefficient ODEs. Classical ODE quantities like the Hamiltonian and Wrońskian may then be utilized. As applications, we obtain a Frobenius theorem and establish new Pohožaev identities. We also give a detailed proof for the non-degeneracy of the fast-decay singular solution of the fractional Lane–Emden equation.

AMS subject classifications: Primary 35J61; Secondary 35R11, 53A30.
Key words: ODE methods, non-local equations, fractional Hardy operators, Frobenius theorem.

1 Introduction

Let $\gamma\in(0,1)$. We consider radially symmetric solutions the fractional Laplacian equation

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$$(-\Delta)^\gamma u = Au^p \quad \text{in } \mathbb{R}^n\setminus\{0\}, \tag{1.1}$$

with an isolated singularity at the origin. Here

$$p\in\left(\frac{n}{n-2\gamma},\frac{n+2\gamma}{n-2\gamma}\right],$$

and the constant $A:=A_{n,p,\gamma}(>0)$ is chosen so that $u_0(r)=r^{-\frac{2\gamma}{p-1}}$ is a singular solution to the equation. Note that this the exact growth rate around the origin of any other solution with non-removable singularity according to [5,11,18].

Note that for

$$p=\frac{n+2\gamma}{n-2\gamma}$$

the problem is critical for the Sobolev embedding $W^{\gamma,2}\hookrightarrow L^{\frac{2n}{n-2\gamma}}$. In addition, for this choice of nonlinearity, the equation has good conformal properties and, indeed, in conformal geometry it is known as the fractional Yamabe problem. In this case the constant A coincides with the Hardy constant $\Lambda_{n,\gamma}$ given in (1.7).

There is an extensive literature on the fractional Yamabe problem by now. See [35,36, 40,44] for the smooth case, [3,7,22,23] in the presence of isolated singularities, and [4,5,34] when the singularities are not isolated but a higher dimensional set.

In this paper we take the analytical point of view and study several non-local ODE that are related to problem (1.1), presenting both survey and new results, in the hope that this paper serves as a guide for non-local ODE. A non-local equation such as (1.1) for radially symmetric solutions $u=u(r)$, $r=|x|$, requires different techniques than regular ODE. For instance, existence and uniqueness theorems are not available in general, so one cannot reduce it to the study of a phase portrait. Moreover, the asymptotic behavior as $r\to 0$ or $r\to\infty$ is not clear either.

However, we will show that, in some sense, (1.1) behaves closely to its local counterpart (the case $\gamma=1$), which is given by the second order ODE

$$\partial_{rr}u+\frac{n-1}{r}\partial_r u = Au^p.$$

In particular, for the survey part we will extract many results for non-local ODEs from the long paper [5] but without many of the technicalities. However, from the time since [5] first appeared, some of the proofs have been simplified; we present those in detail.

The main underlying idea, which was not fully exploited in [5], is to write problem (1.1) as an infinite dimensional ODE system. Each equation in the system is a standard second order ODE, the non-locality appears in the coupling of the right hand sides (see Corollary 4.2). The advantage of this formulation comes from the fact that, even though we started with a non-local ODE, we can still use a number of the standard results, as long as one takes care of this coupling. For instance, we will be able to write the indicial roots