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Global Well-Posedness for the 2-D MHD Equations with Magnetic Diffusion

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Abstract. In this paper, we consider the 2-D MHD equations with magnetic resistivity but without dissipation on the torus. We prove that if the initial data is small in $H^4(\mathbb{T}^2)$, then the 2-D MHD equations are globally well-posed. To our knowledge, this is the first global well-posedness result for this system.

AMS subject classifications: 35Q35, 35B30

Key words: MHD equations, globally well-posedness.

1 Introduction

In this paper, we study the 2-D MHD equations with magnetic resistivity but without dissipation:

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nabla p=b\cdot\nabla b, \\ \partial_t b-\Delta b+u\cdot\nabla b=b\cdot\nabla u, \\ \nabla\cdot u=\nabla\cdot b=0, \\ u(0)=u_0(x), \quad b(0)=b_0(x). \end{cases} \tag{1.1}$$

Here $u(t,x)$ is the velocity field, $b(t,x)$ is the magnetic field and $p(t,x)$ is the pressure. When $b=0$, it is reduced to the classical incompressible Euler equations.

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It is well-known that the 2-D MHD equations with dissipation and magnetic resistivity are globally well-posed. Thanks to the beautiful structure of the vorticity ω

$$\partial_t \omega + u\cdot\nabla\omega = 0$$

the maximum principle gives $\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}$, thus, the 2-D incompressible Euler equations are also globally well-posed for smooth data. However, the question of whether smooth solution of the 2-D ideal MHD equations develops a singularity in a finite time is a challenging problem. In fact, this problem is also open even for the 2-D MHD equations with partial diffusion. Let us refer to [13] for more mathematical results similar to the Navier-Stokes (or Euler) equations.

For the system (1.1), there holds the basic energy law

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \|\nabla b(s)\|_{L^2}^2 ds = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

Furthermore, we introduce $\omega = \partial_{x_1} u^2 - \partial_{x_2} u^1$ and $j = \partial_{x_1} b^2 - \partial_{x_2} b^1$. Using the vorticity formulation of the system (1.1), one can prove that

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^t \|\nabla j(s)\|_{L^2}^2 ds \le C\left(\|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2\right). \tag{1.2}$$

Based on the above estimates, it is easy to prove the global existence of weak solution for the system (1.1) via compactness method [9]. However, it remains open whether weak solution is smooth and unique. In [4], Cao and Wu established the following regularity and unique criterion of weak solution:

$$\sup_{q\ge 2} \frac{1}{q}\int_0^T \|\nabla u(t)\|_{L^q} dt < +\infty.$$

This condition is slightly weaker than the classical Beale-Kato-Majda condition

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < +\infty.$$

Therefore, the estimate (1.2) is far away from the one required by the criterion.

Cao and Wu [4] considered the following 2-D anisotropic MHD equations

$$\begin{cases} \partial_t u - \nu_1 \partial_{x_1}^2 u - \nu_2 \partial_{x_2}^2 u + u\cdot\nabla u + \nabla p = b\cdot\nabla b, \\ \partial_t b - \eta_1 \partial_{x_1}^2 b - \eta_2 \partial_{x_2}^2 b + u\cdot\nabla b = b\cdot\nabla u. \end{cases}$$

They proved the global well-posedness for the following two cases: $\nu_1 > 0$, $\eta_2 > 0$ or $\nu_2 > 0$, $\eta_1 > 0$. However, it remains open for the other two cases: $\nu_1 > 0$, $\eta_1 > 0$ or $\nu_2 > 0$, $\eta_2 > 0$, see [3] for further results.