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# On Nonnegative Solution of Multi-Linear System with Strong $\mathcal{M}_z$-Tensors

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**Abstract.** A class of structured multi-linear system defined by strong $\mathcal{M}_z$-tensors is considered. We prove that the multi-linear system with strong $\mathcal{M}_z$-tensors always has a nonnegative solution under certain condition by the fixed point theory. We also prove that the zero solution is the only solution of the homogeneous multi-linear system for some structured tensors, such as strong $\mathcal{M}$-tensors, $\mathcal{H}^+$-tensors, strictly diagonally dominant tensors with positive diagonal elements. Numerical examples are presented to illustrate our theoretical results.

**AMS subject classifications**: 15A18, 15A69

**Key words**: $\mathcal{M}_z$-tensor, multi-linear system, nonnegative solution, $\mathcal{M}$-tensor, tensor equation, fixed point theory.

## 1. Introduction

A basic problem in both pure and applied mathematics is solving various kinds of equations. Those like linear system, Sylvester and Riccati equations are well-known [15]. As a generalization of the matrix to higher-order case, tensor has received considerable attention in recent years. Let $\mathcal{A}$ be an $m$-th order $n$-dimensional tensor in $\mathbb{C}^{n\times n\times\cdots\times n} := \mathbb{C}^{[m,n]}$ and vector $\mathbf{b}\in\mathbb{C}^n$, then the linear system could be generalized to a higher-order case represented by tensors, that is,

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}, \tag{1.1}$$

where $\mathcal{A}\in\mathbb{R}^{[m,n]}$ and the left-hand side $\mathcal{A}\mathbf{x}^{m-1}$ is a vector with entries

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$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2}\cdots x_{i_m}$$

for all $i = 1,\ldots,n$, [25]. The multi-linear system (1.1) arises in various applications, such as numerical partial differential equations [12], tensor complementarity problems [6,7], data mining [18] and so on.

Up to now, the multi-linear system (1.1) has been studied by many researchers. If $\mathcal{A}$ is an $\mathcal{M}$-tensor (mainly symmetric or strong $\mathcal{M}$-tensors [11,39]), some results corresponding to (1.1) have been presented; see [12,14,17,20,30,36]. Wang *et al.* considered this problem in a more general case, like with $\mathcal{H}^+$-tensors [29] or nonsingular tensors [32]. Besides, some other type equations, like tensor absolute value equation [13], sparse nonnegative tensor equations [18], $\mathcal{H}$-tensor equation [33], are also studied by researchers. More details could be found in the monographs [9,26,27,34] and the references therein.

Many different definitions on eigenvalues of tensors are proposed and studied recently, such as $M$-eigenvalues [22] and $C$-eigenvalues [37] and so on. One of the most well-known is the $Z$-eigenvalue given by Qi [25] and Lim [19] independently. Recently, Mo *et al.* [23] gave a new class of tensors called (strong) $\mathcal{M}_z$-tensors based on $Z$-eigenvalues and proved that an even-order (strong) $\mathcal{M}$-tensor must be an (strong) $\mathcal{M}_z$-tensor and the converse is not true in general. Some results about positive semi-definiteness and co-positivity are also given.

Motivated by these works, we consider the multi-linear system (1.1) with strong $\mathcal{M}_z$-tensor and we call it strong $\mathcal{M}_z$-tensor equation. Before giving the definition of (strong) $\mathcal{M}_z$-tensor, let us see the definition of $Z$-eigenvalue first.

**Definition 1.1** ([19,25])**.** We call $\lambda$ as a $Z$-eigenvalue of tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, if there exists a nonzero real vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}, \quad \mathbf{x}^\top\mathbf{x} = 1.$$

Such an $\mathbf{x}$ is called a $Z$-eigenvector associated with $\lambda$, and $(\lambda,\mathbf{x})$ is called a $Z$-eigenpair.

The $Z$-spectral radius of a tensor is defined as

$$\rho_z(\mathcal{B}) := \sup\left\{|\lambda| : \lambda \text{ is } Z\text{-eigenvalue of } \mathcal{B}\right\}.$$

By using the $Z$-spectral radius, the definition of $\mathcal{M}_z$-tensor [23] is recalled as follows. And it is a generalization of the $M$-matrix to the high-order case and also contains the even-order $\mathcal{M}$-tensor as a proper subset.

**Definition 1.2.** ([23, Definition 3.1]). Assume that $m$ is even. A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called an $\mathcal{M}_z$-tensor, if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $s \geq \rho_z(\mathcal{B})$ such that

$$\mathcal{A} = s\mathcal{I}_z - \mathcal{B},$$