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A CIP-FEM for High-Frequency Scattering Problem with the Truncated DtN Boundary Condition

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Abstract. A continuous interior penalty finite element method (CIP-FEM) is proposed to solve high-frequency Helmholtz scattering problem by an impenetrable obstacle in two dimensions. To formulate the problem on a bounded domain, a Dirichlet-to-Neumann (DtN) boundary condition is proposed on the outer boundary by truncating the Fourier series of the original DtN mapping into finite terms. Assuming the truncation order $N \geq kR$, where k is the wave number and R is the radius of the outer boundary, then the H^j-stabilities, $j=0,1,2$, are established for both original and dual problems, with explicit and sharp estimates of the upper bounds with respect to k. Moreover, we prove that, when $N \geq \lambda kR$ for some $\lambda > 1$, the solution to the DtN-truncation problem converges exponentially to the original scattering problem as N increases. Under the condition that k^3h^2 is sufficiently small, we prove that the preasymptotic error estimates for the linear CIP-FEM as well as the linear FEM are $C_1kh+C_2k^3h^2$. Numerical experiments are presented to validate the theoretical results.

AMS subject classifications: 65N12, 65N30, 78A40, 78A45
Key words: Helmholtz equation, high-frequency, DtN operator, CIP-FEM, wave-number-explicit estimates.

1 Introduction

In this paper, we study the exterior scattering problem of high-frequency acoustic waves by an impenetrable bounded obstacle in two dimensions. To solve the problem numerically, one should truncate the unbounded domain into a bounded one and impose accurate artificial boundary condition on the truncation boundary. For this purpose, some

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non-reflecting boundary conditions are proposed, such as the perfectly matched layer (PML) boundary condition [4], the Dirichlet-to-Neumann (DtN) boundary condition or transparent boundary condition (TBC) [12], etc. For scattering problems of moderate frequency, there are extensive studies on the PML in the literature (see e.g. [5, 8, 21]). For TBCs of scattering problems, the authors refer to the recent papers [2, 19, 20, 22].

The DtN boundary condition, or equivalently the DtN operator, is expressed by an infinite trigonometric or spherical harmonic series. In [7], Chandler and Monk studied the high-frequency Helmholtz scattering problem with exact DtN boundary condition. They proved the inf-sup condition for the sesquilinear form $b(\cdot,\cdot)$ of weak formulation with inf-sup constant being $\mathcal{O}(k^{-1})$, where k is the wave number. In practice, the DtN operator must be truncated into the sum of a finite N terms. This leads to an approximate problem with the truncated DtN boundary condition. Obviously, using fewer terms makes the numerical methods more efficient. A major task is to prove the well-posedness of the approximate problem and that the approximate solution u^N converges exponentially to the original solution u as $N\to+\infty$.

Based on numerical experiences and heuristic analysis for Helmholtz eigenvalues on the computational domain, Harari and Hughes suggested to take $N\geq kR$ in the truncated DtN operator to enhance the solvability and accuracy of the approximate problem [14], where R is the radius of the computational domain. In [13], the authors circumvented the difficulty of uniqueness by modifying the truncated DtN operator to eliminate real eigenvalues of the Helmholtz problem. In [15], the authors proved that the approximate solution u^N converges algebraically to the exact solution u provided that $N\geq N_0$ for some N_0 large enough. Recently, Xu and Yin proved that u^N converges to u at an exponential rate q^N for some $0<q<1$ under the same condition that $N\geq N_0$ [29]. Since the papers are focused on moderate-frequency problems, both the factor q and the dependence of N_0 on the wave number k are not given explicitly. In [20], Jiang, Li and Zheng first proposed an adaptive finite element method with truncated DtN operator for solving scattering problems. The a posteriori error estimates consist of the residuals from finite element (FE) discretization and the error estimate from the truncation of the DtN operator. In [19], the authors proved that the error estimate for the DtN truncation decays exponentially as $N\to+\infty$. However, there still is a question left: how do the well-posedness of the truncated problem and the convergence rate of u^N to u depend explicitly on the wave number k? The first objective of the paper is to give an attempt for the answer.

The second objective of the paper is to study the continuous interior penalty finite element method (CIP-FEM) for solving the truncated problem. Preasymptotic FE error estimates will be given by emphasizing their explicit dependence on k. The Helmholtz equation with large wave number is highly indefinite. It is well-known that traditional finite element method (FEM) suffers from the effect of pollution errors [1, 16, 17]. For the Helmholtz equation with exact DtN boundary condition, Melenk and Sauter [26] proved that the linear FE error estimate is $\|u-u_h^{\rm FEM}\|_{H^1}\leq C_1kh+C_2k^3h^2$ provided that k^2h is sufficiently small. For impedance boundary condition and PML boundary condition, Wu et al. proved the same error estimate by assuming that k^3h^2 is small enough [11, 23, 28, 30].