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MULTIPLICITY OF POSITIVE SOLUTIONS TO A CLASS OF MULTI-POINT BOUNDARY VALUE PROBLEM*†

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Abstract

In this paper, an existence criterion of multiple positive solutions to a class of nonlinear multi-point boundary value problem is established by using the Guo-Krasnoselskii's fixed-point theorem. At the same time, an example is also included to testify our conclusion.

Keywords positive solutions; multi-point boundary value problem; fixed point

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1 Introduction

Boundary value problems for ordinary differential equations have extensive applications in the fields of physics, chemistry, biology and so on. For example, the deformations of an elastic beam in elastic mechanics can be modeled by a fourth-order boundary value problem. In the past decades, based on some fixed point theorems [1–4], the lower and upper solutions method [5] and some degree theories [6], the existence of positive solutions to boundary value problems for fourth-order and higher order [2–5, 7–12] has been always the focus of investigation, especially the existence of multiple positive solutions. In 2003, Ma [2] and Li [3] established the existence and multiplicity of positive solutions to some fourth-order boundary value problems, respectively. Recently, Zhou [11] investigated the existence and multiplicity of positive solutions to a fourth-order three-point eigenvalue problem by the Guo-Krasnoselskii's fixed-point theorem. For other meaningful results, one can be referred to [7, 9–11].

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In 2014, by using a fixed point theorem on cone and combining with the constant variation method, Kong [12] obtained some results on the existence of at least one positive solution to the following fourth-order multi-point boundary value problem containing parameters

$$\begin{cases} u^{(4)}(t) - \rho^4 u(t) = f(t, u(t)), \\ u(0) = 0, \quad u(1) - \alpha u(\eta) = 0, \\ u''(0) = 0, \quad u''(1) - \alpha u''(\eta) = -\lambda. \end{cases} \tag{1.1}$$

But to the best of our knowledge, there are not any results on the multiple positive solutions to the boundary value problem (1.1). Motivated by the ideas from literatures [2,3,11], our purpose in this paper is to investigate the multiplicity of positive solutions to (1.1), where α is a positive number, $0 < \eta < 1$ satisfies $\alpha\eta < 1$, $\lambda > 0$ and ρ is a parameter satisfying $0 < \rho < \frac{\pi}{2}$.

This paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we state and prove the main result on the existence of multiple positive solutions to (1.1). At the same time, an example is also included to testify the obtained results.

2 Preliminaries

For convenience, we first state some definitions, notations and preliminary results. Throughout this paper, we make the following assumption:

(H_1) $f \in C([0,1] \times [0, +\infty), [0, +\infty))$ and $0 < \int_0^1 f(t, u(t))\mathrm{d}t < +\infty$.

Definition 2.1[12] If a function $u(t)$ satisfies the BVP (1.1), $u(t) \in C^3[0,1] \cap C^4[0,1]$, and $u(t) > 0$ for any $t \in (0,1)$, then the function $u(t)$ is said to be a positive solution to the BVP (1.1).

Theorem 2.1[1] (Guo-Krasnoselskii's fixed-point theorem) *Let E be a Banach space, $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow K$ be a completely continuous operator such that either*

(i) $\|Tu\| \geq \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|,\ u \in K \cap \partial\Omega_2$; or

(ii) $\|Tu\| \leq \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|,\ u \in K \cap \partial\Omega_2$

holds, then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let $C[0,1]$ be endowed with the maximum norm

$$\|u\| = \max_{t \in [0,1]} |u(t)|.$$

Let $G_1(t,s)$ and $G_2(t,s)$ be the Green's functions of the following boundary value problems

$$\begin{cases} u''(t) + \rho^2 u(t) = -v(t), \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$