Commun. Math. Res.
doi: 10.4208/cmr.2020-0052

Vol. **36**, No. 3, pp. 336-353
August 2020

M-Eigenvalues of the Riemann Curvature Tensor of Conformally Flat Manifolds

Yun Miao[1], Liqun Qi[2] and Yimin Wei[3,*]

[1] *School of Mathematical Sciences, Fudan University, Shanghai 200433, P. R. of China.*
[2] *Department of Applied Mathematics, the Hong Kong Polytechnic University, Hong Kong.*
[3] *School of Mathematical Sciences and Shanghai Key Laboratory of Contemporary Applied Mathematics, Fudan University, Shanghai 200433, P. R. of China.*

Received 6 May 2020; Accepted 28 May 2020

Abstract. We investigate the M-eigenvalues of the Riemann curvature tensor in the higher dimensional conformally flat manifold. The expressions of M-eigenvalues and M-eigenvectors are presented in this paper. As a special case, M-eigenvalues of conformal flat Einstein manifold have also been discussed, and the conformal the invariance of M-eigentriple has been found. We also reveal the relationship between M-eigenvalue and sectional curvature of a Riemannian manifold. We prove that the M-eigenvalue can determine the Riemann curvature tensor uniquely. We also give an example to compute the M-eigentriple of de Sitter spacetime which is well-known in general relativity.

AMS subject classifications: 15A48, 15A69, 65F10, 65H10, 65N22

Key words: M-eigenvalue, Riemann curvature tensor, Ricci tensor, conformal invariant, canonical form.

*Corresponding author. *Email addresses:* `ymwei@fudan.edu.cn`, `yimin.wei@gmail.com` (Y. Wei), `15110180014@fudan.edu.cn` (Y. Miao), `maqilq@polyu.edu.hk` (L. Qi)

1 Introduction

The eigenvalue problem is a very important topic in tensor computation [4]. Xiang et al. [18, 19] considered the M-eigenvalue problem of elasticity tensor and discussed the relation between strong ellipticity condition of a tensor and its M-eigenvalues, and then extended the M-eigenvalue problem to the Riemann curvature tensor.

Qi et al. [15] introduced the M-eigenvalues for the elasticity tensor. The M-eigenvalues θ of the fourth-order tensor E_{ijkl} are defined as follows:

$$\begin{cases} E_{ijkl}y^j x^k y^l = \theta x_i, \\ E_{ijkl}x^i y^j x^k = \theta y_l \end{cases}$$

under constraints $x^\top x = 1$ and $y^\top y = 1$. This kind of eigenvalue is closely related to the strong ellipticity and positive definiteness of the material.

Recently, Xiang et al. [19] introduced the M-eigenvalue for the Riemann curvature tensor as follows:

$$R^i_{jkl} y^j x^k y^l = \lambda x^i,$$

where $x^\top x = 1$ and $y^\top y = 1$. They calculated several special cases such as two-dimensional and three-dimensional cases to examine what the M-eigenvalues are. They found that in the low dimensional cases, the M-eigenvalues are closely related to the Gaussian curvature and scalar curvature according to the expression of the Riemann curvature tensor, the Ricci curvature tensor and Theorem Egregium of Gauss. The case of constant curvature has also been considered.

The intrinsic structures of manifolds are important to the differential geometry. In the last century, many results and theorems have been developed by Chern. We further discuss the M-eigenvalue of the Riemann curvature tensor according to the intrinsic geometry structure, and cover Xiang et al. results [19] in the conformal flat manifolds.

This paper is organized as follows. In the preliminaries, we list the formulae and results in the Riemannian geometry, including the symmetries of the Riemann curvature tensor, and give the definition of the M-eigenvalue problem of the Riemann curvature tensor. In the main part of our paper, we first review the result by Xiang et al. [19] in two-dimensional manifold and three-dimensional manifold cases, then we introduce the Ricci decomposition of the Riemann curvature tensor and give the definition of a conformal flat manifold. For conformal flat manifold of dimension m, the Riemann curvature tensor has an explicit expression which is related to the scalar curvature and the Ricci curvature tensor. Using this expression, we give the M-eigenvalue of higher dimensional cases