Commun. Math. Res.
doi: 10.4208/cmr.2020-0012

Vol. **36**, No. 3, pp. 272-295
August 2020

Effective Boundary Conditions for the Heat Equation with Interior Inclusion

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Received 19 March 2020; Accepted 19 May 2020

Abstract. Of concern is the scenario of a heat equation on a domain that contains a thin layer, on which the thermal conductivity is drastically different from that in the bulk. The multi-scales in the spatial variable and the thermal conductivity lead to computational difficulties, so we may think of the thin layer as a thickless surface, on which we impose "effective boundary conditions"(EBCs). These boundary conditions not only ease the computational burden, but also reveal the effect of the inclusion. In this paper, by considering the asymptotic behavior of the heat equation with interior inclusion subject to Dirichlet boundary condition, as the thickness of the thin layer shrinks, we derive, on a closed curve inside a two-dimensional domain, EBCs which include a Poisson equation on the curve, and a non-local one. It turns out that the EBCs depend on the magnitude of the thermal conductivity in the thin layer, compared to the reciprocal of its thickness.

AMS subject classifications: 35K05, 35B40, 35B45, 74K15

Key words: Heat equation, effective boundary conditions, weak solution, a priori estimates, asymptotic behavior.

1 Introduction

This paper is motivated by the following scenario: diffusion/heat conduction occurs on a domain (body) which contains thin layers either near the boundary

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or included in the interior; the thin layers have a drastically different property so that the diffusion rate/thermal conductivity in the layers is different in size and in nature (anisotropy vs. isotropy) than in the bulk. Such a diffusion equation is cumbersome to solve numerically because of the multi-scales in both the spatial variable and the parameters. On the other hand, these diffusion equations are common in applications such as thermal barrier coatings (TBCs) for turbine engine blades, roads in nature reserves enhancing the spreading of animal species, busy commercial pathways accelerating epidemics, a cell's outer and inner membranes, etc. To ease the difficulties resulting from the afore-mentioned multi-scales, one thinks of the thin layers as thickless surfaces and imposes "effective boundary conditions" (EBCs). Formal derivation of EBCs was recorded as early as 1959 in the book of Carslaw and Jaeger [2]; rigorous derivation was initiated by Sanchez-Palencia [16], and Brezis *et al.* [1]. Later on, EBCs of the Poisson equation and heat/diffusion equation were investigated in a series of works [3,5,7–9,11,12,14,15]; see also the review paper [17].

This paper focuses on deriving EBCs for the case when the thin layer is in the interior of a 2-dimensional body, which can be mathematically described as $\Omega=\overline{\Omega_1}\cup\overline{\tilde{\Omega}}\cup\Omega_2\subset\mathbb{R}^2$ (see Fig. 1), where both Ω and Ω_1 are fixed and bounded domains with C^2-smooth boundaries $\partial\Omega$ and Γ_1, respectively, and the thin layer $\tilde{\Omega}$ is uniformly thick with thickness δ, which is obtained as follows. Assume Γ_1 has length ℓ and is parametrized by the arc-length variable $s\in[0,\ell]$ as $\mathbf{p}:[0,\ell)\to\Gamma_1$ in the clockwise fashion. For all small $\delta>0$, define a mapping

$$(s,r)\in[0,\ell)\times(-\delta,\delta)\mapsto x=X(s,r)=\mathbf{p}(s)+r\mathbf{n}(s)\in\mathbb{R}^2,$$

where $\mathbf{n}(s)$ is the unit outer normal vector of Γ_1 at $\mathbf{p}(s)$.

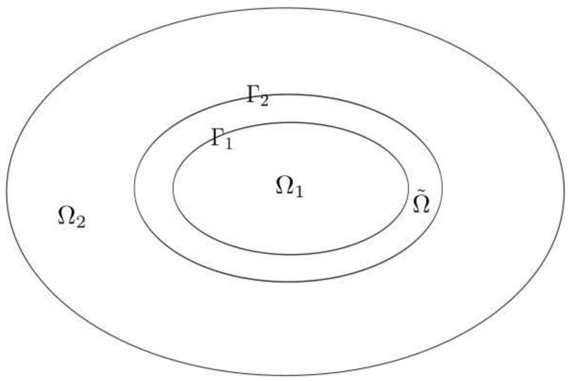

Figure 1: $\Omega=\overline{\Omega_1}\cup\overline{\tilde{\Omega}}\cup\Omega_2$. Ω_1 is fixed and the inclusion layer $\tilde{\Omega}$ is uniformly thick with thickness δ. As $\delta\to0$, $\Gamma_2\to\Gamma_1$.