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Moving-Water Equilibria Preserving HLL-Type Schemes for the Shallow Water Equations

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Abstract. We construct new HLL-type moving-water equilibria preserving upwind schemes for the one-dimensional Saint-Venant system of shallow water equations with nonflat bottom topography. The designed first- and second-order schemes are tested on a number of numerical examples, in which we verify the well-balanced property as well as the ability of the proposed schemes to accurately capture small perturbations of moving-water steady states.

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Key words: Shallow water equations, Harten-Lax-Van Leer (HLL) scheme, well-balanced method, steady-state solutions (equilibria), moving-water and still-water equilibria.

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1 Introduction

In this paper, we develop a new numerical method for the Saint-Venant system of shallow water equations. The Saint-Venant system is a hyperbolic system of balance laws, which was developed in [11] and still used in a wide variety of applications related to modeling water flows in rivers, canals, lakes, coastal areas and even in deep oceans in the situations in which the horizontal length scale is much greater than the vertical length scale. In the one-dimensional (1-D) case, the Saint-Venant system reads as

$$\begin{cases} h_t+(hu)_x=0, \\ (hu)_t+\left(hu^2+\frac{g}{2}h^2\right)_x=-ghB_x, \\ B_t=0, \end{cases} \tag{1.1}$$

where x and t are the spatial and temporal variable, respectively, $h=h(x,t)$ is the water depth, $u=u(x,t)$ is the velocity, $B=B(x)$ is the bottom topography assumed to be time-independent, and g is acceleration due to gravity.

The Jacobian of the system (1.1) has three eigenvalues $\lambda_\pm(h,u)=u\pm\sqrt{gh}$ and $\lambda_0=0$. Therefore, the system (1.1) is hyperbolic as long as $h\geq 0$ and is strictly hyperbolic if $\lambda_+\lambda_-\neq 0$. There are three possible flow regimes depending on the above eigenvalues: (i) if $\lambda_-\lambda_+<0$ then the flow is *subcritical*, (ii) if $\lambda_+\lambda_->0$ then the flow is *supercritical*, (iii) if $\lambda_+\lambda_-=0$ then the flow is *critical*. It is easy to show that the system (1.1) admits the following equilibria:

$$q:=hu\equiv\text{Const}, \quad E:=\frac{u^2}{2}+g(h+B)\equiv\text{Const}, \tag{1.2}$$

where q and E denote the discharge and total energy, respectively. The steady states (1.2) are of great practical importance as many physically relevant water waves are, in fact, their small perturbations. "Lake at rest" steady states or still-water equilibria given by

$$u\equiv 0, \quad h+B\equiv\text{Const}, \tag{1.3}$$

form a subclass of (1.2). When $u\not\equiv 0$, the steady states (1.2) are called moving-water equilibria.

It is well-known that a good numerical method for the Saint-Venant system (1.1) should be capable of exactly preserving all of the equilibria given by (1.2) or at least the still-water equilibria (1.3). Such schemes are called *well-balanced* as they respect a delicate balance between the flux and source terms in the discharge