Regularized DPSS Preconditioners for Singular Saddle Point Problems

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Abstract. Recently, Cao proposed a regularized deteriorated positive and skew-Hermitian splitting (RDPSS) preconditioner for the non-Hermitian nonsingular saddle point problem. In this paper, we consider applying RDPSS preconditioner to solve the singular saddle point problem. Moreover, we propose a two-parameter accelerated variant of the RDPSS (ARDPSS) preconditioner to further improve its efficiency. Theoretical analysis proves that the RDPSS and ARDPSS methods are semi-convergent unconditionally. Some spectral properties of the corresponding preconditioned matrices are analyzed. Numerical experiments indicate that better performance can be achieved when applying the ARDPSS preconditioner to accelerate the GMRES method for solving the singular saddle point problem.

1. Introduction

Consider the singular saddle point problem

$$\mathcal{A}z \equiv \begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1.1}$$

where the matrix $A \in \mathbb{C}^{n \times n}$ is non-Hermitian positive definite, $B \in \mathbb{C}^{n \times m}$ is a rectangular matrix with $rank(B) = r$, $f \in \mathbb{C}^n$ and $g \in \mathbb{C}^m$ are given vectors, with $n \geq m$, B^* denotes the conjugate transpose of B. When $r = m$, the coefficient matrix $\mathcal{A}$ is nonsingular and the linear system (1.1) has an unique solution. When $r < m$, i.e., B is a rank-deficient matrix, then the coefficient matrix $\mathcal{A}$ is singular. Moreover, we

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assume that the linear system (1.1) is consistent, so the solutions of (1.1) exists. In this paper, we focus on the non-Hermitian singular saddle point problem. Background information of saddle point problems in scientific and engineering applications can be found in [5, 15–17] and so on.

Many iteration methods have been proposed for solving the singular saddle point problem (1.1). For instance, Chao *et al.* in [14] studied convergence of P-regular splitting methods for non-Hermitian positive semi-definite linear systems, Zheng *et al.* in [28], Zhang *et al.* in [26] and Zhang *et al.* in [25] improved the Uzawa method, Cao *et al.* in [9] first present a shift-splitting preconditioner for saddle point problems and Cao *et al.* [13] extend generalized shift-splitting iteration method for singular nonsymmetric saddle point problems. Shen *et al.* in [22] studied a class of generalized shift-splitting preconditioners for both nonsingular and singular generalized saddle point problems. Moreover, some variations of the SOR method can be found in [19, 29]. Benzi *et al.* in [6] proposed to apply the HSS iteration method in [4] to solve the saddle point linear system, and then HSS-type methods were analyzed in more detail [1, 3, 7, 12, 23]. Preconditioned GMRES methods were studied in [11, 24, 27]. Especially, when A is non-Hermitian, Liang *et al.* in [20] proposed a two-parameter PDPSS method and the corresponding preconditioner for solving non-Hermitian singular saddle point problems.

In [21], Pan *et al.* put forward a deteriorated positive and skew-Hermitian splitting (DPSS) preconditioner for the non-Hermitian nonsingular saddle point problem. Recently, by adding a regularization Hermitian positive semi-definite matrix, Cao presented a regularized DPSS (RDPSS) iteration method and the corresponding RDPSS preconditioner in [10]. It is proved that the RDPSS iteration method converges unconditionally to the unique solution of saddle point problems. Note that the RDPSS iteration method can be viewed as a valid variant of the RHSS iteration method in [2] for the Hermitian nonsingular saddle point problem. For the singular saddle point problem, RDPSS itetation method reduces to RHSS iteration method when A is Hermitian and Chao *et al.* in [32] proved RHSS is unconditional semi-convergence. In the present paper, we use the RDPSS preconditioner to solve the non-Hermitian singular saddle point problem. In the process of analysis, we prove that the method is unconditionally semi-convergent. The RDPSS method can be used as an efficient preconditioner for Krylov subspace methods, such as the GMRES method. Spectral properties of the corresponding preconditioned matrix are analyzed. Moreover, to further improve the efficiency of the RDPSS preconditioner, the ARDPSS iteration method and corresponding preconditioner are presented and analyzed. Numerical experiments show the ARDPSS preconditioner shows better performance than the RDPSS preconditioner when accelerating the GMRES method.

For sake of simplicity, we use the following notation in this article. The notation 0 denotes a zero matrix of suitable dimension. For a number a_0, the $Re(a_0)$ and $Im(a_0)$ denote the real and imaginary parts of a_0. Respectively, $\mathcal{N}(\cdot)$, $(\cdot)^{-1}$, $(\cdot)^*$, $(\cdot)^T$, $rank(\cdot)$, $\sigma(\cdot)$ and $\rho(\cdot)$ denote the null space, the inverse, the conjugate transpose, the transpose, the rank, the spectrum and the spectral radius of a matrix.