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Newton Linearized Methods for Semilinear Parabolic Equations

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Abstract. In this study, Newton linearized finite element methods are presented for solving semi-linear parabolic equations in two- and three-dimensions. The proposed scheme is a one-step, linearized and second-order method in temporal direction, while the usual linearized second-order schemes require at least two starting values. By using a temporal-spatial error splitting argument, the fully discrete scheme is proved to be convergent without time-step restrictions dependent on the spatial mesh size. Numerical examples are given to demonstrate the efficiency of the methods and to confirm the theoretical results.

AMS subject classifications: 65M10, 78A48

Key words: Newton linearized methods, unconditional convergence, Galerkin FEMs, semilinear parabolic equations.

1. Introduction

In this paper, we present a Newton linearized finite element method as well as its unconditionally optimal error estimate for the following semi-linear parabolic equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u), & x \in \Omega, \quad 0 < t \leq T, \\ u(x,0) = u_0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^d$, ($d = 2$ or 3) is a convex and bounded polygon in R^2 or polyhedron in R^3, $u(x,t)$ is an unknown function defined in $\Omega \times [0,T]$, and $f(u) \in C^2(R)$ is a nonlinear function. Those semi-linear parabolic equations are widely used to model much natural

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phenomena in mechanics, thermodynamics and optics [1,2]. Thus, the equations have attracted plenty of researchers in theoretical and numerical analysis, e.g., [3–15].

The most widely used the second-order numerical schemes for the time-discretization of the semi-linear parabolic equations are the linearized Crank-Nicolson method [16–25],implicit-explicit multi-step methods [26–28], linearized BDF methods [29–32] and so on. In actual applications, these linearized schemes require at least two starting values. One is obtained by the initial value and the other is obtained by some iterative methods or some additional numerical schemes. As a result, some additional computational cost is needed by using the previous mentioned numerical methods.

In the present paper, we present a Newton linearized finite element method for numerically solving the semi-linear parabolic equations, which only needs one start value. Moreover, the fully discrete scheme is proved to be unconditionally convergent. Such convergent result implies that the error estimate holds without certain time-step restrictions dependent on the spatial mesh size. Our inspiration of the proof comes from the recent temporal-spatial error splitting argument, which was firstly proposed by Li and Sun in [33,34] and was successfully applied to analyse some classical time-dependent PDEs [35–39] and some time fractional reaction-diffusion equations [40–42]. We remark that the unconditional convergence result can also be obtained by combing the present Newton linearized method and the other spacial discretization, e.g., mixed finite element method, finite difference method and so on.

We construct the rest of our paper as follows. In Section 2, we establish the Newton linearized FEM schemes for solving Eq. (1.1) and give main convergent results. In Section 3, we prove the convergence of our methods. In Section 4, we give some numerical studies that demonstrate our theoretical convergence results. Finally, we conclude our paper in Section 5.

2. The Newton linearized FEM and main results

In this section, the two-level Newton linearized Galerkin finite element schemes and main results are given.

Following the standard FEM discretization [43], let $\mathcal{T}_h$ be a conforming and shape regular simplicial triangulation or tetrahedra of Ω and let $h = \max_{K\in\mathcal{T}_h}\{\text{diam } K\}$ be the mesh size. Denote V_h is the finite-dimensional subspace of $H_0^1(\Omega)$, which is made of continuous piecewise polynomials of degree r $(r \geq 1)$ on $\mathcal{T}_h$. Let $\tau = T/N$ be time step, where N is a fixed integer. Denote $t_n = n\tau$, $t_{n-\frac{1}{2}} = \frac{1}{2}(t_n + t_{n-1})$ and $u^m = u(x, t_m)$. For a sequence of functions $\{\phi^n\}$, $n = 1, 2, \cdots, N$, we write

$$D_\tau\phi^n = \frac{\phi^n - \phi^{n-1}}{\tau}, \quad \bar{\phi}^n = \frac{1}{2}(\phi^n + \phi^{n-1}). \tag{2.1}$$

With the above notations, a Newton linearized Galerkin FEM is to find $U_h^n \in V_h$ such