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Regularity to a Kohn-Laplace Equation with Bounded Coefficients on the Heisenberg Group

Junli Zhang, Pengcheng Niu*and Xiuxiu Wang

Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710129, China.

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Abstract. In this paper, we concern the divergence Kohn-Laplace equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(X_j^*(a^{ij}X_iu)+Y_j^*(b^{ij}Y_iu)\Big)+Tu=f-\sum_{i=1}^{n}\Big(X_i^*f^i+Y_i^*g^i\Big)$$

with bounded coefficients on the Heisenberg group $\mathbb{H}^n$, where $X_1,\cdots,X_n,Y_1,\cdots,Y_n$ and T are real smooth vector fields defined in a bounded region $\Omega\subset\mathbb{H}^n$. The local maximum principle of weak solutions to the equation is established. The oscillation properties of the weak solutions are studied and then the Hölder regularity and weak Harnack inequality of the weak solutions are proved.

AMS subject classifications: 35B50, 35B65

Key words: Heisenberg group, Kohn-Laplace equation, local maximum principle, Hölder regularity, weak Harnack inequality.

1 Introduction

Under the assumption that the coefficients in the divergence elliptic equation

$$Lu=-\sum_{i,j=1}^{n}D_i(a^{ij}D_ju)=0,\ \ x\in\mathbb{R}^n$$

are bounded measurable, De Giorgi ([8]) proved that the bounded weak solution must be locally Hölder continuous, and deduced the priori estimate of the corresponding Hölder modulus. Nash ([29]) obtained independently this type of estimates of the solutions to

*Corresponding author. *Email addresses:* `jlzhang@mail.nwpu.edu.cn` (J. Zhang), `pengchengniu@nwpu.edu.cn` (P. Niu), `18821781582@163.com` (X. Wang)

the parabolic equation, but the method used is different. Afterwords, Moser ([28]) discovered a new iteration method (later called the Moser iteration) and established the above results for both elliptic equation and parabolic equation with this method.

In recent years, more attention has been paid to the regularity of degenerate partial differential equations (including elliptic equations, parabolic equations and many specific degenerate elliptic equations). Extension of classical regularity theory to degenerate partial differential equations composed of vector fields is an important issue and obtains the substantial development (see [2, 3, 6, 9, 12, 20]). The $C^{1,\alpha}$ regularity of solutions to the p-Laplace equation in the Heisenberg group $\mathbb{H}^1$ was considered by Ricciotti [30], Domokos [10], and Domokos and Manfredi [11]. Dong and Niu studided nondiagonal quasilinear degenerate elliptic systems and gained regularity for weak solutions in [13]. Du, Han and Niu obtained interior Morrey estimates and Hölder continuity for weak solutions to degenerate equations with drift on homogenous groups in [14]. Hou, Feng and Cui proved global Hölder estimates for hypoelliptic operators with drift on homogeneous groups in [22]. Bramanti and Zhu gained L^p and Schauder estimates for nonvariational operators structured on Hörmander vector fields with drift in [4]. Austin and Tyson work with the smoothness of solutions to the operator

$$L_c=-\frac{1}{4}\sum_{i=1}^{n}\left(X_j^2+Y_j^2\right)+cT$$

(c is a real number) in [1]. The aim of this paper is to establish the Hölder regularity of weak solutions to the divergence Kohn-Laplace equation with bounded coefficients on the Heisenberg group.

More concretely, we consider the divergence Kohn-Laplace equation

$$X_j^*(a^{ij}(\xi)X_iu)+Y_j^*(b^{ij}(\xi)Y_iu)+Tu=f-\left(X_i^*f^i+Y_i^*g^i\right) \tag{1.1}$$

with bounded coefficients in $\mathbb{H}^n$, where Ω is a bounded domain in the Heisenberg group $\mathbb{H}^n$ and $X_j^*(a^{ij}(\xi)X_iu)+Y_j^*(b^{ij}(\xi)Y_iu)$ is a degenerate elliptic operator (when $a^{ij}(\xi)=b^{ij}(\xi)$ $=1$, the operator is the known sub-Laplace operator), and Tu can be regarded as a drift term. Here we used the summation convention in (1.1), so that (1.1) can be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(X_j^*(a^{ij}(\xi)X_iu)+Y_j^*(b^{ij}(\xi)Y_iu)\right)+Tu=f-\sum_{i=1}^{n}\left(X_i^*f^i+Y_i^*g^i\right), \tag{1.2}$$

where

$$X_i=\frac{\partial}{\partial x_i}+2y_i\frac{\partial}{\partial t},\quad Y_i=\frac{\partial}{\partial y_i}-2x_i\frac{\partial}{\partial t},\quad i=1,\cdots,n;\ \ T=\frac{\partial}{\partial t}.$$

X_j^* and $Y_j^*\,(j=1,\cdots,n)$ are the adjoint operators of X_j and $Y_j\,(j=1,\cdots,n)$ respectively (see [2]):

$$X_j^*=-\frac{\partial}{\partial x_j}-2y_j\frac{\partial}{\partial t},\quad Y_j^*=-\frac{\partial}{\partial y_j}+2x_j\frac{\partial}{\partial t}.$$